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REMARKABLE RELATIONS BETWEEN THE CENTRAL BINOMIAL SERIES, EULERIAN POLYNOMIALS, AND POLY-BERNOULLI NUMBERS, LEADING TO STEPHAN'S OBSERVATION

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Abstract. The central binomial series at negative integers are expressed as a linear combination of values of certain two polynomials. We show that one of the polynomials is a special value of the bivariate Eulerian polynomial and the other polynomial is related to the anti-diagonal sum of poly-Bernoulli numbers. As an application, we prove Stephan's observation from 2004.

1. Introduction

The *central binomial series* is a Dirichlet series defined by

$$\zeta_{\mathrm{CB}}(s) = \sum_{n=1}^{\infty} \frac{1}{n^s \binom{2n}{n}} \quad (s \in \mathbb{C}). \tag{1.1}$$

Borwein *et al* [**6**] studied special values $\zeta_{\mathrm{CB}}(k)$ at positive integers and recovered some remarkable connections. A classical evaluation is

$$\zeta_{\mathrm{CB}}(4) = \frac{17\pi^4}{3240} = \frac{17}{36}\zeta(4).$$

In particular, for $k \geq 2$, Borwein *et al* showed that $\zeta_{\mathrm{CB}}(k)$ can be written as a $\mathbb{Q}$-linear combination of multiple zeta values and multiple Clausen and Glaisher values.

On the other hand, Lehmer [**15**] proved that, for $k \leq 1$, $\zeta_{\mathrm{CB}}(k)$ is a $\mathbb{Q}$-linear combination of 1 and $\pi/\sqrt{3}$. For example, we have

$$\zeta_{\mathrm{CB}}(1) = \frac{1}{3}\frac{\pi}{\sqrt{3}}, \quad \zeta_{\mathrm{CB}}(0) = \frac{1}{3} + \frac{2}{9}\frac{\pi}{\sqrt{3}},$$
$$\zeta_{\mathrm{CB}}(-1) = \frac{2}{3} + \frac{2}{9}\frac{\pi}{\sqrt{3}}, \quad \zeta_{\mathrm{CB}}(-2) = \frac{4}{3} + \frac{10}{27}\frac{\pi}{\sqrt{3}}.$$

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He considered the general sum

$$\sum_{n=1}^{\infty} \frac{(2x)^{2n}}{n\binom{2n}{n}} = \frac{2x \arcsin(x)}{\sqrt{1-x^2}} \quad (|x| < 1)$$

and its derivatives to derive interesting series evaluations. More precisely, Lehmer provided the following explicit formula for the special values $\zeta_{\mathrm{CB}}(k)$ at negative integers. Define two sequences of polynomials $(p_k(x))_{k\geq -1}$ and $(q_k(x))_{k\geq -1}$ by the initial values $p_{-1}(x) = 0$, $q_{-1}(x) = 1$ and the recursion

$$\begin{aligned} p_{k+1}(x) &= 2(kx+1)p_k(x) + 2x(1-x)p_k'(x) + q_k(x), \\ q_{k+1}(x) &= (2(k+1)x+1)q_k(x) + 2x(1-x)q_k'(x). \end{aligned} \tag{1.2}$$

Then, for $k \geq -1$, we have

$$\sum_{n=1}^{\infty} \frac{(2n)^k (2x)^{2n}}{\binom{2n}{n}} = \frac{x}{(1-x^2)^{k+3/2}} \Big(x\sqrt{1-x^2}\, p_k(x^2) + \arcsin(x) q_k(x^2) \Big). \tag{1.3}$$

Consequently,

$$\zeta_{\mathrm{CB}}(-k) = \frac{1}{3}\left(\frac{2}{3}\right)^k p_k\left(\frac{1}{4}\right) + \frac{1}{3}\left(\frac{2}{3}\right)^{k+1} q_k\left(\frac{1}{4}\right)\frac{\pi}{\sqrt{3}} \in \mathbb{Q} + \mathbb{Q}\frac{\pi}{\sqrt{3}}. \tag{1.4}$$

The first few polynomials are: $p_0(x) = 1$, $p_1(x) = 3$, $p_2(x) = 8x + 7$; and $q_0(x) = 1$, $q_1(x) = 2x + 1$, $q_2(x) = 4x^2 + 10x + 1$.

In 2004, Stephan in the OEIS [**23**, A098830] observed that the rational part of (1.4) is nothing but (a third of) a sum of poly-Bernoulli numbers of negative indices. Poly-Bernoulli numbers $B_n^{(k)}$ are a generalization of classical Bernoulli numbers using polylogarithm functions and were introduced by Kaneko [**12**]. We will give a precise definition in Section 3.

CONJECTURE 1.1. (Stephan's conjecture, stated by Kaneko [**13**]) *For any $n \geq 0$,*

$$\left(\frac{2}{3}\right)^n p_n\left(\frac{1}{4}\right) = \sum_{k=0}^{n} B_{n-k}^{(-k)}.$$

In this article, we connect both polynomials $p_n(x)$ and $q_n(x)$ to known numbers and polynomials. More precisely, we prove Stephan's conjecture (relating this way $p_n(x)$ to the poly-Bernoulli numbers) using the fact that the polynomial sequence $q_n(x)$ is a generalization of the classical Eulerian polynomials.

2. The polynomials $q_n(x)$ and bivariate Eulerian polynomials

Eulerian polynomials were studied by Euler himself. Since then they have been studied and became classical. Several extensions, generalizations, and applications are known today.

Let $\mathfrak{S}_n$ denote the set of permutations $\pi = \pi_1\pi_2\cdots\pi_n$ of $[n] = \{1, 2, \ldots, n\}$. For each $\pi \in \mathfrak{S}_n$, the excedance set is defined as $\mathrm{Exc}(\pi) = \{i \in [n] \mid \pi_i > i\}$. We set $\mathrm{exc}(\pi) = |\mathrm{Exc}(\pi)|$. It is well known that the *Eulerian number* $A(n, k)$ counts the number of permutations $\pi \in \mathfrak{S}_n$ with $\mathrm{exc}(\pi) = k$. For instance, $A(3, 1) = 4$ because there are four permutations of $\{1, 2, 3\}$ with $\mathrm{exc}(\pi) = 1$, namely, 132, 213, 312, 321. A map $f\colon \mathfrak{S}_n \to \mathbb{Z}_{\geq 0}$

satisfying $|\{\pi \in \mathfrak{S}_n \mid f(\pi) = k\}| = A(n, k)$ is often called an *Eulerian statistic*. The map exc is an example of Eulerian statistics. By Foata's fundamental transformation, it is also known that the number of permutations with k excedances is the same as the number of permutations with k descents, or, equivalently formulated, with $k + 1$ ascending runs (see Bóna's book [**5**]).

The *Eulerian polynomial* is defined by

$$A_n(x) = \sum_{\pi \in \mathfrak{S}_n} x^{\mathrm{exc}(\pi)} = \sum_{k=0}^{n-1} A(n, k) x^k.$$

The generating function of the Eulerian polynomials is given as

$$\sum_{n=0}^{\infty} A_n(x) \frac{t^n}{n!} = \frac{1 - x}{e^{t(x-1)} - x}.$$

For a more detailed history and properties on the Eulerian numbers (polynomials) and Eulerian statistics, the articles [**5, 9, 19**] are good references.

We recall now a generalization of the Eulerian polynomial introduced by Foata and Schützenberger [**9**, Chapter IV-3]. Here we define a shifted version. Let $\mathrm{cyc}(\pi)$ denote the number of cycles in the disjoint cycle representation of $\pi \in \mathfrak{S}_n$.

Definition 2.1. (Bivariate Eulerian polynomial) For any integer $n \geq 0$, let $F_0(x, y) = 1$ and define

$$F_n(x, y) = \sum_{\pi \in \mathfrak{S}_n} x^{\mathrm{exc}(\pi)} y^{\mathrm{cyc}(\pi)} \quad (n > 0).$$

Example 2.2. We have $F_3(x, y) = y^3 + 3xy^2 + x^2y + xy$ as follows from Table 1.

TABLE 1. Permutations in $\mathfrak{S}_3$ with their weights.

$\mathfrak{S}_3$	123 = (1)(2)(3)	132 = (1)(23)	213 = (12)(3)	231 = (123)	312 = (132)	321 = (13)(2)
$\mathrm{exc}(\pi)$	0	1	1	2	1	1
$\mathrm{cyc}(\pi)$	3	2	2	1	1	2

The generating function of the bivariate Eulerian polynomials is given by

$$\mathcal{F}(x, y; t) := \sum_{n=0}^{\infty} F_n(x, y) \frac{t^n}{n!} = \left(\frac{1 - x}{e^{t(x-1)} - x} \right)^y. \tag{2.1}$$

Savage and Viswanathan [**22**] derived several identities for the polynomials. Here we recall their recursion formula.

PROPOSITION 2.3. *For $n \geq 0$,*

$$F_{n+1}(x, y) = \left(x(1 - x) \frac{d}{dx} + nx + y \right) F_n(x, y),$$

with the initial value $F_0(x, y) = 1$.

Note that by the definition, we have $F_n(x, 1) = A_n(x)$. Moreover, for $y = r \in \mathbb{Z}_{\geq 2}$, the polynomials $F_n(x, r)$ are the r-Eulerian polynomials originally studied by Riordan [**20**]. In addition, we have $F_{n+1}(x, -1) = -(x-1)^n$ and $F_{n+1}(1, y) = y(y+1)\cdots(y+n)$ for any $n \geq 0$.

The surprising fact is, however, that the values at $y = 1/k$, for any positive integer k, have also nice combinatorial interpretations. Namely, for a sequence $\mathbf{s} = (s_i)_{i\geq 1}$ of positive integers, let the **s**-*inversion sequence* of length n be defined as

$$I_n^{(\mathbf{s})} = \{(e_1, \dots, e_n) \in \mathbb{Z}^n \mid 0 \leq e_i < s_i \text{ for } 1 \leq i \leq n\}.$$

The ascent statistic on $e \in I_n^{(\mathbf{s})}$ is

$$\mathrm{asc}(e) = \left| \left\{ 0 \leq i < n \colon \frac{e_i}{s_i} < \frac{e_{i+1}}{s_{i+1}} \right\} \right|,$$

with the convention that $e_0/s_0 = 0$. Then the **s**-*Eulerian polynomials* are defined by

$$E_n^{(\mathbf{s})}(x) = \sum_{e \in I_n^{(\mathbf{s})}} x^{\mathrm{asc}(e)}.$$

For more properties of the **s**-Eulerian polynomials, see also Savage and Visontai [**21**]. Note that for $\mathbf{s} = (i)_{i\geq 1} = (1, 2, 3, \dots)$, the **s**-Eulerian polynomials are the classical Eulerian polynomials, $E_n^{(\mathbf{s})} = A_n(x)$. Savage and Viswanathan [**22**] showed that for $\mathbf{s} = ((i-1)k + 1)_{i\geq 1} = (1, k+1, 2k+1, 3k+1, \dots)$, where k is a positive integer, it holds that

$$E_n^{(\mathbf{s})}(x) = k^n F_n\left(x, \frac{1}{k}\right).$$

They called the coefficients in this special case the $1/k$-*Eulerian numbers*. The $1/k$-Eulerian numbers play a role in the theory of k-lecture hall polytopes [**22**] and enumerate certain statistics in k-Stirling permutations [**17**]. We now show that, for $k = 2$, the $E_n^{(1,3,5,\dots)}(x) = 2^n F_n(x, 1/2)$ is the same as the $q_n(x)$ polynomial sequence in Lehmer's identity as Table 2 suggests.

TABLE 2. The polynomials $2^n F_n(x, 1/2)$ and $q_n(x)$.

n	$2^n F_n(x, 1/2)$	$q_n(x)$
-1	—	1
0	1	1
1	1	$2x+1$
2	$2x+1$	$4x^2+10x+1$
3	$4x^2+10x+1$	$8x^3+60x^2+36x+1$
4	$8x^3+60x^2+36x+1$	$16x^4+296x^3+516x^2+116x+1$
5	$16x^4+296x^3+516x^2+116x+1$	$\cdots$

THEOREM 2.4. *The generating function*

$$Q(x, t) := \sum_{n=0}^{\infty} q_{n-1}(x) \frac{t^n}{n!}$$

equals $\mathcal{F}(x, 1/2; 2t)$, *that is,* $q_n(x) = 2^{n+1} F_{n+1}(x, 1/2)$ *for any* $n \geq -1$.

Proof. By translating the recursion in (1.2), the generating function $Q(x, t)$ is characterized by the differential equation

$$\left((2xt-1)\frac{d}{dt} + 2x(1-x)\frac{d}{dx} + 1\right)Q(x, t) = 0$$

and the initial condition $Q(x, 0) = 1$. We can check that the function

$$\left(\frac{1-x}{e^{2t(x-1)} - x}\right)^{1/2} = \mathcal{F}(x, 1/2; 2t) = \sum_{n=0}^{\infty} 2^n F_n(x, 1/2)\frac{t^n}{n!}$$

satisfies these conditions by a direct calculation. □

The relation between the polynomials $F_n(x, 1/2)$ and $q_n(x)$ shed light on a proof of Stephan's conjecture, which follows in the next section.

3. The polynomials $p_n(x)$ and a proof of Stephan's conjecture

In this section, we focus on the polynomial sequence $p_n(x)$ in the expression of Lehmer (1.3). We prove the observation of Stephan, who noticed a relation of the sequence with the poly-Bernoulli numbers. Poly-Bernoulli numbers were introduced by Kaneko [**12**] by the polylogarithm function ($\mathrm{Li}_k(z) = \sum_{m=1}^{\infty} z^m/m^k$ for any integer k) as a generalization of the classical Bernoulli numbers. The *poly-Bernoulli numbers* $B_n^{(k)} \in \mathbb{Q}$ are defined by

$$\sum_{n=0}^{\infty} B_n^{(k)}\frac{t^n}{n!} = \frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}}.$$

Poly-Bernoulli numbers have attractive properties. In particular, the values with negative indices k enumerate several combinatorial objects (see, for instance, [**3, 4, 8, 11**] and references therein).

As one of the most basic properties, Arakawa and Kaneko [**1**] showed that

$$\sum_{k=0}^{n}(-1)^k B_{n-k}^{(-k)} = 0$$

holds for any positive integer n. Since then, several authors have generalized the formula for the alternating anti-diagonal sum in [**14, 18**], but not much is known about the anti-diagonal sum in Conjecture 1.1.

In most of the combinatorial interpretations, the roles of n and k are separately significant, hence it is not natural to consider the anti-diagonal sum. However, one of the interpretations, where this is natural, is the set of permutations with *ascending-to-max property* [**10**]. A permutation $\pi \in \mathfrak{S}_n$ is called *ascending-to-max* if, for any integer i, $1 \le i \le n-2$:

(a) if $\pi^{-1}(i) < \pi^{-1}(n)$ and $\pi^{-1}(i+1) < \pi^{-1}(n)$ then $\pi^{-1}(i) < \pi^{-1}(i+1)$, and
(b) if $\pi^{-1}(i) > \pi^{-1}(n)$ and $\pi^{-1}(i+1) > \pi^{-1}(n)$, then $\pi^{-1}(i) > \pi^{-1}(i+1)$.

In other words: record a permutation in one-line notation and draw an arrow from value i to $i+1$ for each i. Then, the permutation has the ascending-to-max property if all the arrows starting from the left of n point forward and all the arrows starting from an element to the right of n point backward. For instance, 47518362 has the property, but 41385762 does not. It follows from the results of Bényi and Hajnal [**2**] that the number of permutations $\pi \in \mathfrak{S}_{n+1}$ with the ascending-to-max property is given by the anti-diagonal sum $b_n = \sum_{k=0}^{n} B_{n-k}^{(-k)}$. However, no explicit formula or recursion was known about the sequence b_n.

Our first result is a recursion for the sequence b_n.

PROPOSITION 3.1. *The sequence $(b_n)_{n\geq 0}$ satisfies the recurrence relation $b_0 = 1$ and*

$$3b_{n+1} = 2b_n + \sum_{k=0}^{n} \binom{n+1}{k} b_k + 3. \tag{3.1}$$

In order to prove this theorem, we need some preparations. Recall that by [**1**, p. 163], we have

$$\sum_{n=0}^{\infty} b_n x^n = \sum_{j=0}^{\infty} \frac{(j!)^2 x^{2j}}{(1-x)^2(1-2x)^2 \cdots (1-(j+1)x)^2} = \frac{1}{(1-x)^2} \sum_{j=0}^{\infty} f_j\left(2-\frac{1}{x}, 2-\frac{1}{x}\right), \tag{3.2}$$

where $(x)_j = x(x+1)(x+2)\cdots(x+j-1)$ is the Pochhammer symbol and we put

$$f_j(x, y) = \frac{(j!)^2}{(x)_j (y)_j}.$$

By a direct calculation, we have

$$\begin{aligned}\sum_{n=0}^{\infty}\sum_{k=0}^{n} \binom{n+1}{k} b_k x^n &= \sum_{k=0}^{\infty} b_k x^k \sum_{n=0}^{\infty} \binom{n+k+1}{k} x^n = \sum_{k=0}^{\infty} b_k x^{k-1}\left(\frac{1}{(1-x)^{k+1}} - 1\right)\\ &= \frac{1-x}{x(1-2x)^2} \sum_{j=0}^{\infty} f_j\left(3-\frac{1}{x}, 3-\frac{1}{x}\right) - \frac{1}{x(1-x)^2}\\ &\quad\times \sum_{j=0}^{\infty} f_j\left(2-\frac{1}{x}, 2-\frac{1}{x}\right).\end{aligned}$$

Thus, the desired recursion in (3.1) is equivalent to

$$\frac{2(2-x)}{(1-x)^2} \sum_{j=0}^{\infty} f_j\left(2-\frac{1}{x}, 2-\frac{1}{x}\right) = \frac{3}{1-x} + \frac{1-x}{(1-2x)^2} \sum_{j=0}^{\infty} f_j\left(3-\frac{1}{x}, 3-\frac{1}{x}\right). \tag{3.3}$$

To prove (3.3), we derive a useful equation.

LEMMA 3.2. *For any $j \in \mathbb{Z}_{\geq 0}$, we have*

$$\begin{aligned}&(x-1)(x-2)(f_j(x-2, y) - f_{j-1}(x-2, y)) + (x-1)(2x-5) f_{j-1}(x-1, y)\\ &\qquad - (x-1)(x-y-1) f_j(x-1, y) - (x-2)^2 f_{j-1}(x, y)\\ &= \begin{cases} (x-1)(y-1) & \text{if } j = 0,\\ 0 & \text{if } j > 0,\end{cases}\end{aligned}$$

where we put $f_{-1}(x, y) = 0$.

Proof. By direct calculation, one can verify it. □

Proof of Proposition 3.1. We prove (3.3). Setting $x \to 3 - 1/x$ and $y \to 2 - 1/x$ and applying Lemma 3.2, we obtain

$$f_j\left(2 - \frac{1}{x}, 2 - \frac{1}{x}\right) = \frac{(1-x)^2}{(1-2x)(2-x)} f_j\left(3 - \frac{1}{x}, 2 - \frac{1}{x}\right) - \frac{1-x}{2-x}\left(f_{j+1}\left(1 - \frac{1}{x}, 2 - \frac{1}{x}\right) - f_j\left(1 - \frac{1}{x}, 2 - \frac{1}{x}\right)\right).$$

Summing up both sides over $j = 0, 1, 2, \ldots$, we have

$$\sum_{j=0}^{\infty} f_j\left(2 - \frac{1}{x}, 2 - \frac{1}{x}\right) = \frac{1-x}{2-x} + \frac{(1-x)^2}{(1-2x)(2-x)} \sum_{j=0}^{\infty} f_j\left(3 - \frac{1}{x}, 2 - \frac{1}{x}\right).$$

From Lemma 3.2 again for $x \to 3 - 1/x$ and $y \to 3 - 1/x$, we conclude (3.3). □

Remark 3.3. Unfortunately, we could not provide a combinatorial proof for this recurrence, though it would be very interesting to find one using, for instance, the permutations with the ascending-to-max property.

To relate the sequence $(b_n)_{n\geq 0}$ to the polynomial sequence $(p_n(x))_{n\geq -1}$, we next derive the generating function for $p_n(x)$.

PROPOSITION 3.4. *We have*

$$P(x, t) := \sum_{n=0}^{\infty} p_{n-1}(x)\frac{t^n}{n!} = \frac{e^{(1-x)t}\left(\arcsin(x^{1/2}e^{(1-x)t}) - \arcsin(x^{1/2})\right)}{x^{1/2}(1 - xe^{2(1-x)t})^{1/2}}.$$

Proof. From (1.3), we have

$$\sum_{k=0}^{\infty}\sum_{n=1}^{\infty} \frac{(2n)^{k-1}(2x)^{2n}}{\binom{2n}{n}}\frac{t^k}{k!} = \frac{x}{(1-x^2)^{1/2}}\left(x\sqrt{1-x^2}\,P\left(x^2, \frac{t}{1-x^2}\right) + \arcsin(x)\,Q\left(x^2, \frac{t}{1-x^2}\right)\right).$$

By applying (1.3) with $k = -1$ again, the left-hand side equals

$$\sum_{n=1}^{\infty} \frac{(2n)^{-1}(2xe^t)^{2n}}{\binom{2n}{n}} = \frac{xe^t \arcsin(xe^t)}{(1-x^2e^{2t})^{1/2}}.$$

Thus, combining with Theorem 2.4, we have

$$P\left(x^2, \frac{t}{1-x^2}\right) = \frac{e^t \arcsin(xe^t)}{x(1-x^2e^{2t})^{1/2}} - \frac{\arcsin(x)}{x\sqrt{1-x^2}}\mathcal{F}\left(x^2, \frac{1}{2}; \frac{2t}{1-x^2}\right) = \frac{e^t\left(\arcsin(xe^t) - \arcsin(x)\right)}{x(1-x^2e^{2t})^{1/2}},$$

which implies the claim. □

We define the sequence a_n as special values of $p_n(x)$,

$$a_n = \left(\frac{2}{3}\right)^n p_n\left(\frac{1}{4}\right). \tag{3.4}$$

Using the generating function of $p_n(x)$, we obtain the recurrence formula that the sequence $(a_n)_{n\geq 0}$ satisfies.

PROPOSITION 3.5. *The sequence $(a_n)_{n\geq 0}$ defined in (3.4) satisfies $a_0 = 1$ and*

$$3a_{n+1} = 2a_n + \sum_{k=0}^{n} \binom{n+1}{k} a_k + 3.$$

Proof. By Proposition 3.4, the generating function for a_n is given by

$$\sum_{n=0}^{\infty} a_n \frac{t^{n+1}}{(n+1)!} = \frac{3}{2} P\left(\frac{1}{4}, \frac{2}{3}t\right) = \frac{6e^{t/2}\big(\arcsin(e^{t/2}/2) - \arcsin(1/2)\big)}{(4-e^t)^{1/2}}. \tag{3.5}$$

Since this function satisfies the differential equation

$$\left((4-e^t)\frac{d}{dt} - 2\right)\frac{3}{2}P\left(\frac{1}{4}, \frac{2}{3}t\right) = 3e^t,$$

the coefficients a_n satisfy the desired recurrence formula. □

In conclusion, we have the main theorem.

THEOREM 3.6. *Conjecture 1.1 is true, i.e., for any $n \geq 0$,*

$$\left(\frac{2}{3}\right)^n p_n\left(\frac{1}{4}\right) = \sum_{k=0}^{n} B_{n-k}^{(-k)}.$$

Proof. Proposition 3.1 and Proposition 3.5 imply the theorem. □

In the course of our proof, we obtain two types of generating functions in (3.2) and (3.5) for the sequences $(a_n)_{n\geq 0} = (b_n)_{n\geq 0}$. As a corollary, we have an explicit formula for the anti-diagonal sum, (see [**3**, p. 24]).

COROLLARY 3.7. *One has*

$$b_n = \sum_{k=0}^{n} B_{n-k}^{(-k)} = \frac{(-1)^{n+1}}{2} \sum_{j=1}^{n+1} (-1)^j j! \left\{ {n+1 \atop j} \right\} \frac{\binom{2j}{j}}{3^{j-1}} \sum_{i=0}^{j-1} \frac{3^i}{(2i+1)\binom{2i}{i}}.$$

Proof. The result follows from the explicit formula by Borwein and Girgensohn [**7**] and Theorem 3.6.

As a final remark, we show that the polynomial $p_n(x)$ can also be expressed in terms of bivariate Eulerian polynomials. □

THEOREM 3.8. *For any $n \geq 0$, we have*

$$p_n(x) = 2^n \sum_{k=0}^{n} \binom{n+1}{k} F_{n-k}(x, 1/2) F_k(x, 1/2).$$

Proof. Consider

$$\begin{aligned} P(x,t) &= \frac{e^{(1-x)t}\big(\arcsin(x^{1/2}e^{(1-x)t}) - \arcsin(x^{1/2})\big)}{x^{1/2}(1-xe^{2(1-x)t})^{1/2}} \\ &= \mathcal{F}\left(x, \frac{1}{2}; 2t\right) \frac{1}{x^{1/2}(1-x)^{1/2}} \big(\arcsin(x^{1/2}e^{(1-x)t}) - \arcsin(x^{1/2})\big). \end{aligned}$$

Since

$$\frac{d}{dt}\frac{1}{x^{1/2}(1-x)^{1/2}}\big(\arcsin(x^{1/2}e^{(1-x)t})-\arcsin(x^{1/2})\big)=\mathcal{F}\left(x,\,\frac{1}{2};\,2t\right),$$

it holds that

$$\frac{1}{x^{1/2}(1-x)^{1/2}}\big(\arcsin(x^{1/2}e^{(1-x)t})-\arcsin(x^{1/2})\big)=\sum_{n=0}^{\infty}2^n F_n(x,\,1/2)\frac{t^{n+1}}{(n+1)!}.$$

Thus, we have

$$P(x,\,t)=\sum_{n=1}^{\infty}\left(2^{n-1}\sum_{k=0}^{n-1}\binom{n}{k}F_{n-k-1}(x,\,1/2)F_k(x,\,1/2)\right)\frac{t^n}{n!},$$

which concludes the proof. □

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