http://toc.ui.ac.ir

Transactions on Combinatorics
ISSN (print): 2251-8657, ISSN (on-line): 2251-8665
Vol. 12 No. 1 (2023), pp. 47-63.

www.ui.ac.ir

GENERALIZED BARRED PREFERENTIAL ARRANGEMENTS

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Abstract. We investigate a generalization of Fubini numbers. We present the combinatorial interpretation as barred preferential arrangements with some additional conditions on the blocks. We provide a proof for a generalization of Nelsen's Theorem. We consider these numbers from a probabilistic view point and demonstrate how they can be written in terms of the expectation of random descending factorial involving the negative binomial process.

1. Introduction

A *preferential arrangement* of the set $[n] = \{1, 2, 3, \dots, n\}$ is an ordered partition, i.e., a list of pairwise disjoint non-empty subsets of $[n]$ such that the union of the subsets is $[n]$. The subsets are called *blocks*.

Preferential arrangements are enumerated by the Fubini numbers (ordered Bell numbers, geometric numbers)

$$w_n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} k!,$$

where $\left\{ {n \atop k} \right\}$ denote the Stirling numbers of the second kind.

An interesting fact is that the Fubini numbers, w_n, appear in the evaluation of the series

$$\sum_{k=0}^{\infty} \frac{k^n}{2^k} = 2w_n,$$

and also as the n'th moments of the random variable with geometric distribution [10].

Communicated by Peter Csikva.

MSC(2010): Primary: 05C15; Secondary: 20D60, 13.

Keywords: barred preferential arrangements, generalized Stirling numbers, geometric polynomials, negative binomial process.

Article Type: Research Paper.

Received: 12 August 2021, Accepted: 11 January 2022.

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http://dx.doi.org/10.22108/TOC.2022.130037.1894 .

Fubini numbers are the special values of geometric polynomials,

$$w_n(x) = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} k! x^k,$$

that play important role for instance in combinatorics, analytics and probability theory. For this reason several generalizations and studies from different point of views can be found in different lines of researches. However, the connection between these lines seem to be sometimes lost. Our aim is to provide some combinatorial and probabilistic insight for numbers that arise in some algebraic and analytical motivated generalizations.

Pippenger [24] introduced *barred preferential arrangements with a single bar*. The idea is that if candidates have been interviewed for a position, one might want to separate some who are worthy of being hired from those who are not. This idea can be generalized assuming that there are some ranks into which candidates may be hired. Ahlbach-Usatine-Pippenger [2] studied *barred preferential arrangements* with a given arbitrary λ number of bars.

We obtain *barred preferential arrangement* when we insert λ bars (where $\lambda \in \mathbb{N}_0 = \{0, 1, 2, 3, \dots, \}$) in between (before or after) the blocks of a preferential arrangement. The λ bars induce $\lambda + 1$ *sections* in which the elements are preferentially arranged (see [2, 23]). For example, barred preferential arrangements of the set [6] with two bars and three bars, respectively:

a. 35 2| |1 4 6,
b. |5 136 4 2| |.

The two bars in *a.* give rise to three sections; namely, the first section (from left to right) has two blocks $\{3, 5\}$ and $\{2\}$, the second section is empty, and the third section has three blocks $\{1\}$, $\{4\}$, and $\{6\}$. Similarly, the barred preferential arrangement in *b.* has four sections of which three are empty.

The number of preferential arrangements with a single bar [25, A005649] counts also compatible bipartitional partitions [13]. The number of preferential arrangements is the special value at $x = 1$ of higher order geometric polynomials

$$w_n^{(r)}(x) = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} (r)(r-1)\cdots(r-k+1)x^k, \quad r > 0$$

defined by Boyadzhiev [4].

One of the motivations of this study is to give a combinatorial explanation of an interesting identity conjectured first by Nelsen [15].

Theorem 1.1. [20, Nelsen's Theorem] *For any real number γ and non-negative integer n it holds*

$$\sum_{k=0}^{n} \sum_{s=0}^{k} \binom{k}{s} (-1)^{k-s} (\gamma+s)^n = \frac{1}{2} \sum_{s=0}^{\infty} \frac{(\gamma+s)^n}{2^s}. \tag{1.1}$$

Nelsen-Knuth-Binz-Williams [20] provided three alternative proofs of the identity (1.1). We present in this paper a generalized version of Nelsen's Theorem (1.1) and show how it can be interpreted combinatorially in terms of barred preferential arrangements.

Nelsen-Schmidt [21] introduced the family of generating functions:

$$\frac{e^{\gamma x}}{2-e^x}. \tag{1.2}$$

It is well known that for $\gamma = 0$ the function (1.2) is the generating function of preferential arrangements (see [14, 18]). For $\gamma = 2$ the authors interpreted (1.2) as the generating function of the number of chains in the power set of an n element set. Nelsen and Schmidt posed the question "could there be combinatorial structures associated with either $[n]$ or the power set of $[n]$ whose integer sequences are generated by members of the family in (1.2) for other values of γ?" (We refer to this question as the Nelsen-Schmidt question, and to the generating function in (1.2) as the Nelsen-Schmidt generating function.)

In order to answer the Nelsen-Schmidt question Nkonkobe-Murali [23] investigated the family of functions given for any non-negative integers γ and λ in the form

$$\frac{e^{\gamma x}}{(2-e^x)^\lambda}. \tag{1.3}$$

Nkonkobe-Murali [23] showed that (1.3) enumerates the so called *restricted barred preferential arrangements*. In this paper we study a generalization of (1.3), (hence a generalization of the Nelsen-Schmidt generating function) given as

$$\frac{e^{\gamma x}}{(2-e^{\beta x})^\lambda}, \tag{1.4}$$

where β and γ are non-negative integers and $(\beta, \gamma) \neq (0, 0)$.

Remark 1.2. *We note that the function defined in* (1.4) *are special cases of the* higher order generalized geometric polynomials *introduced by Kargin-Cekim* [17] *as*

$$\sum_{n=0}^{\infty} w_n^\lambda(x;\alpha,\beta,\gamma)\frac{t^n}{n!} = \frac{(1+\alpha t)^{\gamma/\alpha}}{(1-x((1+\alpha t)^{\beta/\alpha}-1))^\lambda}. \tag{1.5}$$

As the name indicates the family of polynomials (1.5) *is also a kind of generalization of the geometric polynomials,* $\frac{1}{2-e^x}$*, that are well studied objects, see for instance* [3, 5, 8, 11, 19]*. A combinatorial study of the higher order generalized geometric polynomials can be found in* [22].

We also study in this paper the probabilistic aspects of the polynomials (1.4). It turns out that we can connect them to the negative binomial process which is defined for $(Z_\lambda(t))_{t\geq 0}$:

$$P(Z_\lambda(t) = j) = \binom{-\lambda}{j}\left(-\frac{t}{t+1}\right)^j\left(\frac{1}{t+1}\right)^\lambda, \quad j \in \mathbb{N}_0. \tag{1.6}$$

We show how (1.4) can be written as the expectation of random descending factorials.

The outline of the paper is as follows. In Section 2, we recall some important facts from the literature, and consider barred preferential arrangements with 1 bar. In Section 3, we present results on the number of barred preferential arrangements with arbitrary many bars. Finally, in Section 4, we generalize the model even more using methods of the probability theory.

http://dx.doi.org/10.22108/TOC.2022.130037.1894

2. Generalized Barred Preferential Arrangements with 2 Sections

In order to reveal the combinatorial properties of the model for the function (1.4) including three parameters, we think it is worth to consider first the model having only two parameters in detail. For this reason we focus in this section on the combinatorial aspects of the functions

$$\frac{e^{\gamma x}}{2-e^{\beta x}}. \tag{2.1}$$

Let $H_n(\beta,\gamma)$ denote the coefficients of $\frac{x^n}{n!}$ in (2.1), i.e.,

$$H_n(\beta,\gamma)=\left[\frac{x^n}{n!}\right]\frac{e^{\gamma x}}{2-e^{\beta x}}.$$

Our goal is to describe a combinatorial interpretation of these numbers, $H_n(\beta,\gamma)$ and to prove identities combinatorially using our interpretation, and so providing elementary and simple proofs for this important class of numbers.

Let $\mathcal{H}_n(\beta,\gamma)$ denote the set of barred preferential arrangements on n elements with one bar (so with two sections), such that the elements of the left hand side are labeled further with a number between $\{1,\ldots,\gamma\}$, while the elements right to the bar with a number from the set $\{1,\ldots,\beta\}$. Clearly, $|\mathcal{H}_n(\beta,\gamma)|=H_n(\beta,\gamma)$.

We want to highlight another aspect, showing the place of the numbers $\mathcal{H}_n(\beta,\gamma)$ in a bigger picture. Hsu-Shiue [16] introduced the *generalized Stirling numbers*, $S(n,k,\alpha,\beta,\gamma)$, as follows.

Definition 2.1. [16] *For $n\geq 1$ an integer, α,β,γ real or complex numbers with $(\alpha,\beta,\gamma)\neq(0,0,0)$ the* generalized Stirling pair $\{S^1,S^2\}=\{S(n,k;\alpha,\beta,\gamma),S(n,k;\beta,\alpha,-\gamma)\}$ *with three parameters are defined by*

$$(t|\alpha)_n=\sum_{k=0}^{n}S^1(n,k)(t-\gamma|\beta)_k \quad \textit{and}$$

$$(t|\beta)_n=\sum_{k=0}^{n}S^2(n,k)(t+\gamma|\alpha)_k,$$

where $(t|\alpha)_n$ denotes the generalized factorial of t with increment α *defined for any integer $n\geq 1$ as*

$$(t|\alpha)_n=t(t-\alpha)\cdots(t-n\alpha+\alpha)$$

and $(t|\alpha)_0=1$.

The generalized Stirling pair includes several special cases, as the classical Stirling number of the first and second kind, r-Stirling numbers of the first and second kind, Lah numbers, r-Lah numbers, Whitney numbers, r-Whitney numbers, Carlitz's degenerate Stirling numbers of both kinds, Howard degenerate Stirling numbers of both kinds and so on (see [16, 17]).

One combinatorial interpretation of the numbers $i!\beta^i S(n,k;\alpha,\beta,\gamma)$ was given by Corcino-Hsu-Tan [9]. This model is a partition such that the blocks have an extra structure, so called *cyclic ordered labeled compartments*. Further, the model is defined from a statistical point of view in the sense that the way a sample is created plays a key role.

Corcino-Hsu-Tan [9] showed that $i!\beta^i S(n,i;\alpha,\beta,\gamma)$ is the number of ways to distribute n distinct balls, one ball at a time into $i+1$ distinct blocks, first i of which has β distinct compartments and a last block with γ distinct compartments such that

(1) the compartments in each block are given cyclic ordered numbering,
(2) the capacity of each compartment is limited to one ball,
(3) each successive α available compartment in a block can only have the leading compartment getting a ball,
(4) the first i blocks are non-empty.

For instance, suppose the first ball lands in the 4th compartment of the 3th block. The next α compartments, i.e., the compartments numbered 5, 6, ..., $\alpha+3$ will be closed. Suppose the second ball lands in compartment $\beta-2$ in the 3th block. Then the compartments $\beta-1$, β, 1, 2, 3, $\alpha+4$, ..., $2\alpha-3$ will be closed and so on.

The number of distributions satisfying properties (1), (2) and (3), but not requiring (4), i.e., the first i cells may be empty is given by $(\beta i+\gamma|\alpha)_n$ [9]. We recall two explicit formulas for the generalized Stirling numbers given in [9].

Lemma 2.2. [9] *For $\alpha,\beta,\gamma\in\mathbb{N}_0$, where $(\alpha,\beta,\gamma)\neq(0,0,0)$, we have*

$$S(n,i,\alpha,\beta,\gamma)=\frac{1}{\beta^i i!}\Delta^i(\beta i+\gamma|\alpha)_n\big|_{s=0},$$

$$S(n,i,\alpha,\beta,\gamma)=\frac{1}{\beta^i i!}\sum_s(-1)^{i-s}\binom{i}{s}(\beta s+\gamma|\alpha)_n.$$

Corcino-Corcino generalized the ordered Bell numbers using the generalized Stirling numbers [7].

Definition 2.3. [7] *For $\alpha,\beta,\gamma\in\mathbb{N}_0$ with $(\alpha,\beta,\gamma)\neq(0,0,0)$ the* generalized Bell numbers *are defined as*

$$B_n(\alpha,\beta,\gamma)=\sum_{i=0}^{n} i!\beta^i S(n,i,\alpha,\beta,\gamma).$$

We also recall the generating function of generalized Bell numbers in Lemma 2.4.

Lemma 2.4. [7] *For real/complex α,β,γ such that $(\alpha,\beta,\gamma)\neq(0,0,0)$,*

$$\sum_{n=0}^{\infty}B_n(\alpha,\beta,\gamma)\frac{x^n}{n!}=\frac{(1+\alpha t)^{\gamma/\alpha}}{2-(1+\alpha t)^{\beta/\gamma}}. \tag{2.2}$$

Remark 2.5. *From the generating function in (2.2) the generating function of $\frac{e^{\gamma x}}{2-e^{\beta x}}$ can be derived.*

The generalization of the series expression is proven in [7]

$$B_n(\alpha,\beta,\gamma)=\frac{1}{2}\sum_{k=0}^{\infty}\frac{(\beta k+\gamma|\alpha)_n}{2^k}. \tag{2.3}$$

In this paper we will only need the cases when $\alpha=0$, so we state explicitly some special cases ($\alpha=\beta=0$ and $\alpha=0$) that follows from the results above.

http://dx.doi.org/10.22108/TOC.2022.130037.1894

Corollary 2.6. *For i and β positive integers, $\beta^i i! S(n, i, 0, \beta, 0)$ is the number of ways of partitioning an n-element set into i non-empty blocks where each of the i blocks has β labeled compartments.*

$$S(n, i, 0, \beta, 0) = \frac{1}{\beta^i i!} \sum_{s=0}^{i} (-1)^{i-s} \binom{i}{s} (\beta s)^n. \tag{2.4}$$

Corollary 2.7. *The number of partitioning $[n]$ into i non-empty blocks with β labeled compartments and a possible empty $(i+1)^{th}$ block with γ labeled compartments is $i!\beta^i S(n, i, 0, \beta, \gamma)$.*

Let us define the following properties.

Property 2.8. *Elements are distributed into i ordered blocks such that each block has β labelled compartments.*

Property 2.9. *Elements are distributed into γ labelled compartments.*

The main result of this section is Theorem 2.10 which answers the Nelsen-Schmidt question in a generalized form.

Theorem 2.10. *The generating function $\frac{e^{\gamma x}}{2-e^{\beta x}}$ for β, γ non-negative integers (where $(\beta, \gamma) \neq (0, 0)$), is that of the number of barred preferential arrangements with one bar such that one section has Property 2.8 and the other section has Property 2.9.*

Proof. Let r be the number of elements contained in the section with Property 1. By (2.4) and Lemma 2.3 the number of preferential arrangements of r elements with Property 1 is $B_r(0, \beta, 0)$. The remaining $n - r$ elements are distributed in the section with Property 2 in γ^{n-r} ways. Hence,

$$H_n(\beta, \gamma) = \sum_{r=0}^{n} \binom{n}{r} B_r(0, \beta, 0) \gamma^{n-r}. \tag{2.5}$$

□

Remark 2.11. *Throughout the remainder of this paper in forming barred preferential arrangements (BPAs) where applicable, the section with Property 2.9 will be the first section from the left (referred to as the* special section*) the remaining sections will all have Property 2.8 unless stated otherwise.*

$H_n(\beta, \gamma)$ can be expressed with the Fubini numbers.

Theorem 2.12.

$$H_n(\beta, \gamma) = \sum_{r=0}^{n} \binom{n}{r} \beta^r \gamma^{n-r} w_r.$$

Proof. By a similar argument as of Theorem 2.10 if the number of elements contained in the section with Property 1 is r, there are $\sum_{i=0}^{r} \beta^r i! \left\{ {r \atop i} \right\} = \beta^r w_r$ ways to construct that part. □

The classical recursion, $w_n = \sum_{j=0}^{n-1} \binom{n}{j} w_j$ of Fubini numbers generalizes as given in Theorem 2.13.

http://dx.doi.org/10.22108/TOC.2022.130037.1894

Theorem 2.13. *$\beta \geq 0$ and $n, \gamma \geq 1$,*

$$H_n(\beta,\gamma) = \gamma^n + \sum_{i=0}^{n-1} \binom{n}{i} H_i(\beta,\gamma)\beta^{n-i}.$$

Proof. We obtain an element of the set $h \in \mathcal{H}_n(\beta,\gamma)$ the following way: if there is no element on the right hand side of the bar, then we need only to assign to each element of $[n]$ a number from $[\gamma]$, which gives γ^n possibilities. If there is at least one element to the right of the bar, then first let us construct the block right next to the bar in this section from $(n-i)$ elements in $\binom{n}{n-i}\beta^{n-i}$ ways. The remainder i elements form an element h_i of $\mathcal{H}_i(\beta,\gamma)$. Since $n-i \neq 0$, we obtain the number of all elements in $\mathcal{H}_n(\beta,\gamma)$ by summing up over i, where i runs from 0 to $n-1$. □

The next recursion is a generalization of a convolution formula.

Theorem 2.14. *For $n, \beta, \gamma \geq 0$, where $(\beta,\gamma) \neq (0,0)$,*

$$H_{n+1}(\beta,\gamma) = \gamma H_n(\beta,\gamma) + \beta \sum_{i=0}^{n} \binom{n}{i} H_i(\beta,\gamma) H_{n-i}(\beta,\beta).$$

Proof. The recursion is based on the process of the inserting the $(n+1)$th element into a barred preferential arrangement on n elements. First, we can insert the $(n+1)$th element into the block of the left section. Then, we just need to choose a label from $[\gamma]$ for this new element. Otherwise, let B^* denote the block into which we add $(n+1)$. Cut the barred preferential arrangement before B^* and let i denote the number of elements in the part before B^*. The first part is then a barred preferential arrangement from $\mathcal{H}_i(\beta,\gamma)$, while the second part can be seen also as a barred preferential arrangement of the rest of the elements with B^* as the special block next to left of the bar, i.e., from $\mathcal{H}_{n-i}(\beta,\beta)$. We choose in $\binom{n}{i}$ ways the elements for the first part, and choose the label of $(n+1)$ in β ways. Multiplying these together and summing up by letting the index i to run, we obtain the theorem. □

Using the classical technique of inclusion-exclusion we can express the generalized Bell numbers with the numbers $H_i(\beta,\gamma)$.

Theorem 2.15. *For $n, \beta, \gamma \geq 0$, $(\beta,\gamma) \neq (0,0)$,*

$$B_n(0,\beta,0) = \sum_{i=0}^{n} \binom{n}{i} H_i(\beta,\gamma)(-1)^{n-i}\gamma^{n-i}.$$

Proof. Let $\mathcal{B}_i$ be the number of barred preferential arrangement of the set $\mathcal{H}_n(\beta,\gamma)$ with at least $(n-i)$ elements in the first, special block with γ compartments. $|\mathcal{B}_i| = \binom{n}{n-i} H_i(\beta,\gamma)\gamma^{n-i}$. The application of the inclusion- exclusion principle completes the proof. □

3. Generalized Barred Preferential Arrangements

In this section we consider the function

$$\frac{e^{\gamma x}}{(2-e^{\beta x})^{\lambda}}, \tag{3.1}$$

http://dx.doi.org/10.22108/TOC.2022.130037.1894

where $\gamma \in \mathbb{N}_0$ and $\lambda, \beta \in \mathbb{N}$ (positive integers).

Let $H_n(\lambda,\beta,\gamma)$ denote the coefficients of $\frac{x^n}{n!}$ in the polynomial (3.1)

$$H_n(\lambda,\beta,\gamma) = \left[\frac{x^n}{n!}\right] \frac{e^{\gamma x}}{(2-e^{\beta x})^{\lambda}}.$$

The combinatorial interpretation of the numbers $H_n(\lambda,\beta,\gamma)$ is given in Theorem 3.1.

Theorem 3.1. *Given $\lambda, \gamma \in \mathbb{N}_0$ such that $(\lambda,\gamma) \neq (0,0)$, and $\beta \in \mathbb{N}$, $H_n(\lambda,\beta,\gamma)$ is the number of barred preferential arrangements of $[n]$ such that λ of the sections have Property 2.8 and one section has Property 2.9.*

Proof. First, we write our function as an infinite sum

$$\frac{1}{2-e^{\beta x}} = \frac{1}{2}\sum_{k=0}^{\infty} \frac{e^{(\beta k)x}}{2^k}$$

Thus, the coefficients are given as

$$\left[\frac{x^n}{n!}\right] \frac{1}{2-e^{\beta x}} = \frac{1}{2}\sum_{k=0}^{\infty} \frac{(\beta k)^n}{2^k}.$$

The identity (2.5) implies

$$H_n(\lambda,\beta,\gamma) = \sum_{r_1+\cdots+r_{\lambda+1}=n} \binom{n}{r_1, r_2, \ldots, r_{\lambda+1}} \gamma^{r_1} \prod_{i=2}^{\lambda+1} B_{r_i}(0,\beta,0). \tag{3.2}$$

□

Remark 3.2. *The special case $\lambda = 1, \beta = 2$, and $\gamma = 0$ on Theorem 3.1 is the* [25, sequence A216794].

Let $\mathcal{H}_n(\lambda,\beta,\gamma)$ denote the set of barred preferential arrangements with λ bars (so $\lambda+1$ sections), such that the first section includes one special block with elements labeled from the set $\{1,\ldots,\gamma\}$, and the elements in the rest of the blocks labeled from the set $\{1,\ldots,\beta\}$. Next we prove some recursions for the numbers $H_n(\lambda,\beta,\gamma) = |\mathcal{H}_n(\lambda,\beta,\gamma)|$.

We express first the numbers $H_n(\lambda,\beta,\gamma)$ using the Stirling numbers of the second kind.

Theorem 3.3. *For $\gamma, \lambda \in \mathbb{N}_0$, where $(\lambda,\gamma) \neq (0,0)$,*

$$H_n(\lambda,\beta,\gamma) = \sum_{r=0}^{n} \beta^r \gamma^{n-r} \sum_{i=0}^{r} \binom{\lambda-1+i}{i} i! \left\{ r \atop i \right\}.$$

Proof. Let r denote the number of elements distributed into the part with Property 1, i.e., into a preferential arrangement with λ sections created by the $\lambda-1$ inserted bars. Construct i blocks in $\left\{ r \atop i \right\}$ ways and order it in $i!$ ways. Now arrange bars and blocks, which can be done in $\binom{\lambda-1+i}{i}$ ways. □

Theorem 3.4. *For $\gamma, \lambda \in \mathbb{N}_0$, where $(\lambda,\gamma) \neq (0,0)$,*

$$H_{n+1}(\lambda,\beta,\gamma) = \gamma H_n(\lambda,\beta,\gamma) + \lambda\beta \sum_{i=0}^{n} \binom{n}{i} H_i(1,\beta,\beta) H_{n-i}(\lambda,\beta,\gamma). \tag{3.3}$$

Proof. We enumerate the set $\mathcal{H}_{n+1}(\lambda,\beta,\gamma)$ based on the position of the element $(n+1)$. It can be included in the special first block, which gives $\gamma H_n(\lambda,\beta,\gamma)$ possibilities. Otherwise, let B^* be the block that contains $(n+1)$. Consider the portion of the barred preferential arrangement from B^* till the next bar to its right (including B^* itself), and let i be the number of elements contained in these blocks. This portion can be seen as a barred preferential arrangement from the set $\mathcal{H}_i(\beta,\beta) = \mathcal{H}_i(1,\beta,\beta)$. Ignoring this portion of the barred preferential arrangements, the remaining elements form a barred preferential arrangements from $\mathcal{H}_{n-i}(\lambda,\beta,\gamma)$. For this construction we need to choose the i elements out of the n elements in $\binom{n}{i}$ ways, the section in that B^* is placed in λ ways and finally, the label of $(n+1)$ in β ways. Multiplying these together and summing up completes the argument. □

Theorem 3.5. *For $\gamma,\lambda \in \mathbb{N}_0$, where $(\lambda,\gamma) \neq (0,0)$,*

$$H_{n+1}(\lambda,\beta,\gamma) = \gamma H_n(\lambda,\beta,\gamma) + \lambda\beta H_n(\lambda+1,\beta,\gamma+\beta). \tag{3.4}$$

Proof. Again, the left hand side is the size of the set $\mathcal{H}_{n+1}(\lambda,\beta,\gamma)$. Consider the $(n+1)$th element. If it is contained in the first section, (let's denote this block by Γ), then there are $\gamma H_n(\lambda,\beta,\gamma)$ possibilities to obtain such a barred preferential arrangement on $n+1$ elements from a one on n elements. Assume now that the $(n+1)$th element is in a block, say B^*, with β compartments. Decompose the section including B^* as $B_1B^*B_2$, where B_1 and B_2 are ordered partitions with the extra structure of having a label for each element from $[\beta]$ on each block. We reorder the parts of this barred preferential arrangement as follows: Move the block B^* to the left of the first block, and merge Γ and B^* into one block. Insert instead of the block B^* a bar between the sequences of blocks B_1 and B_2, and finally, delete $(n+1)$. We obtain this way a barred preferential arrangement on n elements, with $(\lambda+1)$ bars and $(\gamma+\beta)$ compartments in the first, special block. Hence, the number of such barred preferential arrangements is $H_n(\lambda+1,\beta,\gamma+\beta)$. There are two information that we have to keep in track: which β compartment was the $(n+1)$th element assigned to, and which bar is the inserted bar. Hence, we have $\lambda\beta H_n(\lambda+1,\beta,\gamma+\beta)$ as total number of barred preferential arrangements on $n+1$ elements such that the $(n+1)$th element is not in the first, special block. □

Theorem 3.6. *For $\gamma,\lambda \in \mathbb{N}$,*

$$H_n(\lambda,\beta,\gamma+\beta) = 2H_n(\lambda,\beta,\gamma) - H_n(\lambda-1,\beta,\gamma). \tag{3.5}$$

Proof. Consider the set $\mathcal{H}_n(\lambda,\beta,\gamma+\beta)$. In these barred preferential arrangements the elements in the first block are labeled from the set $\{1,2,\ldots,\gamma,\gamma+1,\ldots,\gamma+\beta\}$. The number of such barred preferential arrangements that have only labels from the set $\{1,\ldots,\gamma\}$ is $H_n(\lambda,\beta,\gamma)$. If there is at least one element with a label from $\{\gamma+1,\ldots,\gamma+\beta\}$, then move these elements to the right of the first bar, to create the first block in the ordered partition of the second section. We obtain this way a barred preferential arrangement with γ compartments in the first, special block and at least one block in the second section with β compartments. How many such barred preferential are there? $H_n(\lambda,\beta,\gamma) - H_n(\lambda-1,\beta,\gamma)$, since we need to exclude the barred preferential arrangements that do not have any block in the second section, which are clearly in bijection with barred preferential arrangements with one less, i.e., $(\lambda-1)$ bars. □

Theorem 3.6 is a generalization of [23, Theorem 9]. The next theorem is a generalization of [2, Theorem 1].

Theorem 3.7. *For $\beta \in \mathbb{N}$, and $\lambda \geq 2$,*

$$H_n(\lambda,\beta,\beta) = \frac{1}{2}H_n(\lambda-1,\beta,\beta) + \frac{1}{2\beta(\lambda-1)}\sum_{i=0}^{n}\binom{n}{i}H_{i+1}(\lambda-1,\beta,0)\beta^{n-i}. \tag{3.6}$$

Proof. First, we write the formula in a combinatorially nicer form.

$$2\beta(\lambda-1)H_n(\lambda,\beta,\beta) = \beta(\lambda-1)H_n(\lambda-1,\beta,\beta) + \sum_{i=0}^{n}\binom{n}{i}H_{i+1}(\lambda-1,\beta,0)\beta^{n-i}$$

Consider the set of elements of $\mathcal{H}_n(\lambda,\beta,\beta)$ such that one of the β compartments is colored red and one of the λ bars, except the first one, is marked with a 0 or a 1. We let $\mathcal{H}_n^*(\lambda,\beta,\beta)$ denote the set of the so obtained decorated barred preferential arrangements. The left hand side of the equality enumerates this set. We describe a map, that associates to each decorated preferential arrangement of $\mathcal{H}_n^*(\lambda,\beta,\beta)$ another barred preferential arrangement so that the image of the map is a set enumerated by the right hand side. Consider the label of the chosen bar. If the bar has a 0, delete the bar and insert a block with a single extra $(n+1)$th element. If the bar is labeled by 1, consider what is right next to the left of the bar. If there is a block, insert $(n+1)$ into this block, if it is another bar, delete this bar. In each cases when inserting $(n+1)$, it is also colored red, i.e., receives the same β-compartment that is chosen. The number of barred preferential arrangements that we obtain by deleting a bar, (and not inserting $(n+1)$) is $\beta(\lambda-1)H_n(\lambda-1,\beta,\beta)$, since one β-compartment is still colored red, and we have only $\lambda-1$ with a 1 marked bar left. In the other cases, we obtain a barred preferential arrangement on $n+1$ elements, i.e., elements of the set $\mathcal{H}_{n+1}(\lambda-1,\beta,\beta)$, such that the first special section does not contain the $(n+1)$th element. This is, because the first bar was not marked, hence during the insertion process $(n+1)$ was never put into the section left to the first bar. The number of these barred preferential elements is $\sum_{i=0}^{n}\binom{n}{i}H_{i+1}(\lambda-1,\beta,0)\beta^{n-i}$. We obtain this formula according to the enumeration of the following pairs: choose the $n-i$ elements for the first special block and construct it in β^{n-i} ways. Combine these blocks with barred preferential arrangements on $(i+1)$ elements with $\lambda-1$ sections and empty first, special section, for which we have $H_{i+1}(\lambda-1,\beta,0)$ possibilities. □

Theorem 3.8. *For $\beta \in \mathbb{N}$,and $\lambda \geq 2$,*

$$H_n(\lambda,\beta,0) = \frac{1}{2\beta(\lambda-1)}H_{n+1}(\lambda-1,\beta,0) + \frac{1}{2}H_n(\lambda-1,\beta,0). \tag{3.7}$$

Proof. This proof is similar to that of Theorem 3.7. We rewrite the identity as

$$2\beta(\lambda-1)H_n(\lambda,\beta,0) = \beta(\lambda-1)H_n(\lambda-1,\beta,0) + H_{n+1}(\lambda-1,\beta,0)$$

The left hand side is the number of decorated barred preferential arrangements of $\mathcal{H}_n^*(\lambda,\beta,0)$, (with empty first section). Inserting $n+1$ according to the above rule, the deletion of the marked bar without inserting $n+1$ leads to barred preferential arrangements on n with one β compartment chosen and one of its $\lambda-1$ bars marked. This gives $\beta(\lambda-1)H_n(\lambda-1,\beta,0)$ possibilities. Deleting the marked bar and inserting $(n+1)$ leads to barred preferential arrangements on $n+1$ elements, $\lambda-1$ bars, and empty first section, for which we have $H_{n+1}(\lambda-1,\beta,0)$ possibilities. □

The following theorem offers a generalization of Nelsen's Theorem discussed in (1.1).

Theorem 3.9. *For $\beta,\gamma,\lambda\in\mathbb{R}$, and $n\in\mathbb{N}_0$, where $(\lambda,\gamma)\neq(0,0)$,*

$$\sum_{k=0}^{n}\sum_{s=0}^{k}\binom{k}{s}(-1)^{k-s}H_n(\lambda-1,\beta,\gamma+\beta s)=\sum_{s=0}^{\infty}\frac{H_n(\lambda-1,\beta,\gamma+\beta s)}{2^{s+1}}. \tag{3.8}$$

Proof. We use the analogue argument as Gross [14] in proving Equations 2 and 4.

$$\frac{e^{\gamma x}}{(2-e^{\beta x})^{\lambda}}=\frac{e^{\gamma x}}{(2-e^{\beta x})^{\lambda-1}}\sum_{k=0}^{\infty}(e^{\beta x}-1)^k.$$

This implies

$$\left[\frac{x^n}{n!}\right]\frac{e^{\gamma x}}{(2-e^{\beta x})^{\lambda}}=\sum_{k=0}^{n}\sum_{s=0}^{k}\binom{k}{s}(-1)^{k-s}H_n(\lambda-1,\beta,\gamma+\beta s).$$

Also,

$$\frac{e^{\gamma x}}{(2-e^{\beta x})^{\lambda}}=\frac{1}{2}\frac{e^{\gamma x}}{(2-e^{\beta x})^{\lambda-1}}\sum_{s=0}^{\infty}\frac{e^{xs\beta}}{2^s}.$$

This gives

$$\left[\frac{x^n}{n!}\right]\frac{e^{\gamma x}}{(2-e^{\beta x})^{\lambda}}=\frac{1}{2}\sum_{s=0}^{\infty}\frac{H_n(\lambda-1,\beta,\gamma+\beta s)}{2^s}.$$

□

Finally, we show that the left hand side of the Equation (3.8) is the number $H_n(\lambda,\beta,\gamma)$, hence, the number of barred preferential arrangements.

Theorem 3.10.

$$H_n(\lambda,\beta,\gamma)=\sum_{k=0}^{n}\sum_{s=0}^{k}(-1)^{k-s}\binom{k}{s}H_n(\lambda-1,\beta,\gamma+\beta s). \tag{3.9}$$

Proof. Consider the first, special section with γ compartments and the second section with β compartments as one section. (Ignore the first bar.) One can consider then this merged section as a special section with $\gamma+k\beta$ compartments, where k denotes the number of blocks that were in the second section. The number of such preferential arrangements is given by $H_n(\lambda-1,\beta,\gamma+k\beta)$. Now we apply the inclusion-exclusion principle based on the property how many blocks of the second section were empty. □

As a final remark we mention how the symbolic method [12] interprets the generating function (1.4) formally. The construction that translates to (1.4) is

$$[\mathrm{SET}(\mathcal{X})]^{\gamma}\times[\mathrm{SEQ}([\mathrm{SET}(\mathcal{X})]^{\beta})_{>0}]^{\lambda}.$$

Combinatorially, this is a pair of objects $(\mathcal{O}_1, \mathcal{O}_2)$, where $\mathcal{O}_1$ and $\mathcal{O}_2$ are the following. $\mathcal{O}_1$ is a γ-tuple of sets, that may be empty. $\mathcal{O}_2$ is a tuple of λ non-empty sequences of β-tuples of sets. Equivalently, $\mathcal{O}_2$ is an arrangement of non-empty subsets such that each subset has β compartments. The pair $(\mathcal{O}_1, \mathcal{O}_2)$ is clearly equivalent to the set $\mathcal{H}_n(\lambda, \beta, \gamma)$.

4. Probabilistic Interpretation

The number of barred preferential arrangements $H_n(\lambda, \beta, \gamma)$ considered in the previous section can be further generalized in a natural way from a probabilistic perspective. In fact, for any $\lambda \in \mathbb{N}$, denote by $(Z_\lambda(t))_{t\geq 0}$ the negative binomial process defined as

$$P(Z_\lambda(t) = j) = \binom{-\lambda}{j}\left(-\frac{t}{t+1}\right)^j \left(\frac{1}{t+1}\right)^\lambda, \quad j \in \mathbb{N}_0. \tag{4.1}$$

Let $\tau > 0$ be such that

$$\tau < \log(1 + 1/t). \tag{4.2}$$

Observe that for any $\lambda \in \mathbb{N}$, $t \geq 0$, and $\tau > 0$ satisfying (4.2), we have

$$\mathbb{E}e^{\tau Z_\lambda(t)} = \sum_{j=0}^{\infty} \binom{-\lambda}{j}\left(-\frac{e^\tau t}{t+1}\right)^j \left(\frac{1}{t+1}\right)^\lambda = \frac{1}{(1 - t(e^\tau - 1))^\lambda} < \infty, \tag{4.3}$$

where $\mathbb{E}$ stands for mathematical expectation. Denote by $\mathcal{E}_\tau$ the set of functions $\phi : \mathbb{N}_0 \to \mathbb{R}$ such that

$$|\phi(j)| \leq Ae^{\tau j}, \quad j \in \mathbb{N}_0,$$

where $A > 0$ and $\tau > 0$ satisfies (4.2). For such functions, the following crucial formula was shown in [1, Theorem 8.1]

$$\mathbb{E}\phi(Z_\lambda(t)) = \sum_{j=0}^{\infty} \phi(j) P(Z_\lambda(t) = j) = \sum_{k=0}^{\infty} \binom{\lambda - 1 + k}{k} \Delta^k \phi(0) t^k. \tag{4.4}$$

Finally, the following auxiliary result will be very useful.

Lemma 4.1. *Let $\lambda, \nu \in \mathbb{N}$, $t \geq 0$, and $\phi \in \mathcal{E}_\tau$. Then,*

$$\mathbb{E}\phi(Z_{\lambda+1}(t)) = \frac{1}{\lambda(t+1)} \mathbb{E}\phi(Z_\lambda(t))(Z_\lambda(t) + \lambda) \tag{4.5}$$

and

$$\mathbb{E}\phi(Z_{\lambda+\nu}(t)) = \sum_{j=0}^{\infty} \mathbb{E}\phi(Z_\lambda(t) + j) \binom{\nu - 1 + j}{j} \left(\frac{t}{t+1}\right)^j \left(\frac{1}{t+1}\right)^\nu. \tag{4.6}$$

Proof. From (4.1), we see that

$$P(Z_{\lambda+1}(t) = j) = \frac{j + \lambda}{\lambda(t+1)} P(Z_\lambda(t) = j), \quad j \in \mathbb{N}_0.$$

We therefore have

$$\mathbb{E}\phi(Z_{\lambda+1}(t)) = \sum_{j=0}^{\infty} \phi(j) P(Z_{\lambda+1}(t) = j)$$

http://dx.doi.org/10.22108/TOC.2022.130037.1894

$$= \frac{1}{\lambda(t+1)} \sum_{j=0}^{\infty} \phi(j)(j+\lambda) P(Z_\lambda(t) = j),$$

thus showing (4.5). As follows from (4.3),

$$\mathbb{E} e^{\tau Z_{\lambda+\nu}(t)} = \mathbb{E} e^{\tau Z_\lambda(t)} \mathbb{E} e^{\tau Z_\nu(t)}.$$

By the uniqueness theorem for Laplace transforms, this means that the law of $Z_{\lambda+\nu}(t)$ is the same as the law of $Z_\lambda(t) + Z_\nu(t)$, where the random variables $Z_\lambda(t)$ and $Z_\nu(t)$ are supposed to be independent. Hence, we have from (4.1)

$$\mathbb{E}\phi(Z_{\lambda+\nu}(t)) = \mathbb{E}\phi(Z_\lambda(t) + Z_\nu(t))$$

$$\sum_{j=0}^{\infty} \mathbb{E}\phi(Z_\lambda(t) + j) \binom{\nu - 1 + j}{j} \left(\frac{t}{t+1} \right)^j \left(\frac{1}{t+1} \right)^\nu.$$

This shows (4.6) and completes the proof. □

From now on, we assume that $n \in \mathbb{N}_0$, $\lambda \in \mathbb{N}$, $\beta, \gamma \in \mathbb{R}$ ($\beta \neq 0$), and $t \geq 0$. Let $H_n(\lambda, \beta, \gamma, t)$ denote the expectation value of $(\beta Z_\lambda(t) + \gamma)^n$,

$$H_n(\lambda, \beta, \gamma, t) = \mathbb{E}(\beta Z_\lambda(t) + \gamma)^n. \tag{4.7}$$

The generating function of such numbers is given in the following result.

Theorem 4.2. *We have*

$$\sum_{n=0}^{\infty} H_n(\lambda, \beta, \gamma, t) \frac{x^n}{n!} = \frac{e^{\gamma x}}{(1 - t(e^{\beta x} - 1))^\lambda}, \qquad |x| < \frac{1}{|\beta|} \log(1 + 1/t).$$

Proof. Replacing τ by βx in (4.3) and recalling (4.7), we have

$$\frac{e^{\gamma x}}{(1 - t(e^{\beta x} - 1))^\lambda} = \mathbb{E} e^{x(\beta Z_\lambda(t) + \gamma)}$$

$$= \sum_{n=0}^{\infty} \mathbb{E}(\beta Z_\lambda(t) + \gamma)^n \frac{x^n}{n!} = \sum_{n=0}^{\infty} H_n(\lambda, \beta, \gamma, t) \frac{x^n}{n!}.$$

Note that the interchange of sum with expectation, whenever $|\beta x| \leq \log(1 + 1/t)$, follows from Fubini's theorem. The proof is complete. □

This result shows that the numbers defined in (4.7) extend the numbers $H_n(\lambda, \beta, \gamma)$. More precisely,

$$H_n(\lambda, \beta, \gamma, 1) = H_n(\lambda, \beta, \gamma),$$

giving in this way a probabilistic meaning to such numbers. In addition, such extended numbers can be expressed in terms of the classical Stirling numbers of the second kind $\left\{ {n \atop k} \right\}$, as the following result shows.

Theorem 4.3. *We have*

$$H_n(\lambda,\beta,\gamma,t) = \frac{1}{(t+1)^\lambda}\sum_{j=0}^{\infty}\binom{\lambda-1+j}{j}\left(\frac{t}{t+1}\right)^j(\beta j+\gamma)^n$$
$$= \sum_{r=0}^{n}\binom{n}{r}\beta^r\gamma^{n-r}\sum_{k=0}^{r}\binom{\lambda-1+k}{k}k!\left\{ {r \atop k} \right\}t^k.$$

Proof. The first equality readily follows from (4.1) and (4.7). Applying formula (4.4) to the polynomial $\phi(x) = p_n(x) = (\beta x+\gamma)^n$, we obtain

$$H_n(\lambda,\beta,\gamma,t) = \mathbb{E}p_n(Z_\lambda(t)) = \sum_{k=0}^{n}\binom{\lambda-1+k}{k}\Delta^k p_n(0)t^k, \tag{4.8}$$

since $\Delta^k p_n(0) = 0, k > n$. On the other hand, denote by $I_r(x) = x^r, r \in \mathbb{N}_0$, the rth monomial function. Applying the operator Δ^k to the formula

$$p_n(x) = (\beta x+\gamma)^n = \sum_{r=0}^{n}\binom{n}{r}\beta^r\gamma^{n-r}I_r(x),$$

we see that

$$\Delta^k p_n(0) = \sum_{r=k}^{n}\binom{n}{r}\beta^r\gamma^{n-r}\Delta^k I_r(x) = \sum_{r=k}^{n}\binom{n}{r}\beta^r\gamma^{n-r}k!\left\{ {r \atop k} \right\}. \tag{4.9}$$

Consequently, the second equality in statement of the theorem follows from (4.8) and (4.9) by interchanging the order of summation. □

Theorem 4.3 may be seen as an extension of known identities. In first place, choosing $\lambda = \beta = t = 1$, we obtain

$$H_n(1,1,\gamma,1) = \mathbb{E}(Z_1(1)+\gamma)^n = \frac{1}{2}\sum_{j=0}^{\infty}\frac{(j+\gamma)^n}{2^j} = \sum_{k=0}^{n}\Delta^k\widetilde{p}_n(0), \tag{4.10}$$

where $\widetilde{p}_n(x) = (x+\gamma)^n$. Identity (4.10) is known as Nelsen's Theorem.

In second place, the polylogarithm function of order $-n$ is defined as

$$Li_{-n}(z) = \sum_{j=1}^{\infty} j^n z^j, \qquad 0 \le z < 1.$$

It is well known that

$$Li_{-n}(z) = \sum_{k=0}^{n}k!\left\{ {n+1 \atop k+1} \right\}\left(\frac{z}{1-z}\right)^{k+1}, \qquad 0 \le z < 1. \tag{4.11}$$

Setting $\lambda = \beta = 1$ and $\gamma = 0$ in Theorem 4.3, we have from (4.7)

$$H_n(1,1,0,t) = \mathbb{E}Z_1(t)^n = \sum_{j=1}^{\infty} j^n\left(\frac{t}{t+1}\right)^j\frac{1}{t+1} = \sum_{k=0}^{n}k!\left\{ {n \atop k} \right\}t^k. \tag{4.12}$$

Making the change $z = t/(t+1)$ in (4.12), we obtain identity (4.11).

Finally, using the probabilistic representation given in (4.7), we can derive in an easy way various kinds of identities involving the generalized numbers $H_n(\lambda,\beta,\gamma,t)$, as done in the following two results.

Theorem 4.4. *We have*

$$H_n(\lambda+1,\beta,\gamma,t)=\frac{H_n(\lambda,\beta,\gamma,t)}{t+1}+\frac{1}{\beta\lambda(t+1)}\sum_{i=0}^{n}\binom{n}{i}H_{i+1}(\lambda,\beta,0,t)\gamma^{n-i}.$$

In particular,

$$H_n(\lambda+1,\beta,0,t)=\frac{H_n(\lambda,\beta,0,t)}{t+1}+\frac{1}{\beta\lambda(t+1)}H_{n+1}(\lambda,\beta,0,t).$$

Proof. Applying (4.5) with $\phi(x)=(\beta x+\gamma)^n$, we obtain

$$H_n(\lambda+1,\beta,\gamma,t)-\frac{H_n(\lambda,\beta,\gamma,t)}{t+1}=\mathbb{E}(\beta Z_{\lambda+1}(t)+\gamma)^n-\frac{1}{t+1}\mathbb{E}(\beta Z_{\lambda}(t)+\gamma)^n$$

$$=\frac{1}{\lambda(t+1)}\mathbb{E}(\beta Z_{\lambda}(t)+\gamma)^n Z_{\lambda}(t). \tag{4.13}$$

The right-hand side in (4.13) equals to

$$\frac{1}{\lambda(t+1)}\sum_{i=0}^{n}\binom{n}{i}\beta^i\mathbb{E}Z_{\lambda}(t)^{i+1}\gamma^{n-i}=\frac{1}{\beta\lambda(t+1)}\sum_{i=0}^{n}\binom{n}{i}H_{i+1}(\lambda,\beta,0,t)\gamma^{n-i}.$$

This, together with (4.13), shows the first identity in Theorem 4.4. The second one follows from (4.13) by setting $\gamma=0$. The proof is complete. □

Theorem 4.5. *We have*

$$H_n(\lambda+1,\beta,\gamma+\beta,t)=\sum_{i=0}^{n}\binom{n}{i}H_i(1,\beta,\beta,t)H_{n-i}(\lambda,\beta,\gamma,t).$$

Proof. As in the proof of Lemma 4.1, the law of the random variable $Z_{\lambda+1}(t)$ is the same as the law of $Z_1(t)+Z_{\lambda}(t)$, where the random variables $Z_1(t)$ and $Z_{\lambda}(t)$ are supposed to be independent. Hence,

$$H_n(\lambda+1,\beta,\gamma+\beta,t)=\mathbb{E}(\beta Z_{\lambda+1}(t)+\gamma+\beta)^n$$

$$=\mathbb{E}(\beta Z_1(t)+\beta+\beta Z_{\lambda}(t)+\gamma)^n=\sum_{i=0}^{n}\binom{n}{i}\mathbb{E}(\beta Z_1(t)+\beta)^i\mathbb{E}(\beta Z_{\lambda}(t)+\gamma)^{n-i}$$

$$=\sum_{i=0}^{n}\binom{n}{i}H_i(1,\beta,\beta,t)H_{n-i}(\lambda,\beta,\gamma,t),$$

thus concluding the proof. □

Acknowledgments

The authors wish to thank the anonymous referee for valuable comments and suggestions. The research conducted in this study was partially funded by the following. The National Research Foundation of South Africa under (94986) and Levenstein(959) of Rhodes University. Also, Ministerio de Ciencia, Innovación y Universidades, Project PGC2018-097621-B-I00.

References

[1] J. A. Adell, Differential calculus for linear operators represented by finite signed measures and applications, *Acta Math. Hungar.*, **138** no. 1-2 (2013) 44–82.

[2] C. Ahlbach, J. Usatine and N. Pippenger, Barred Preferential Arrangements, *Electron. J. Combin.*, **20** no. 2 (2013) PP 55.

[3] H. Belbachir and Y. Djemmada, Congruence for geometric polynomials, *Les Annales Rectis*, **6** (2009) 37–44.

[4] K. N. Boyadzhiev, A series transformation formula and related polynomials, *Int. J. Math. Math. Sci.*, **23** (2005) 38–49.

[5] K. N. Boyadzhiev and A. Dil, Geometric polynomials: properties and applications to series with zeta values, *Anal. Math.*, **42** no. 3 (2016) 203–224.

[6] R. B. Corcino, Some theorems on generalized Stirling numbers, *Ars Combin.*, **60** (2001) 273–286.

[7] R. B. Corcino and C. B. Corcino, On generalized Bell polynomials, *Discrete Dyn. Nat. Soc.*, (2011) 21 pp.

[8] C. B. Corcino, R. B. Corcino, B. Cekim, L. Kargin and S. Nkonkobe, A second type of higher order generalized geometric polynomials and higher order generalized Euler polynomials, *Quaest. Math.*, **45** (2020) 1–19.

[9] R. B. Corcino, L. C. Hsu and E. L. Tan, Combinatorial and statistical applications of generalized Stirling numbers, *J. Math. Res. Exposition*, **21** no. 3 (2001) 337–343.

[10] M. E. Dasef and S. M. Kautz, Some sums of some importance, *College Math. J.*, **28** (1997) 52–55.

[11] L. Euler, Institutiones calculi differentialis cum ejus usu in analysi finitorum ac doctrina serierum, Impensis academiae imperialis scientiarum Petropolitanae, (1755). Also, another edition, Ticini: in typographeo Petri Galeatii superiorum permissu (1787). Available online at http://eulerarchive.maa.org//pages/E212.html

[12] P. Flajolet and R. Sedgewick, *Analytic combinatorics*, Cambridge University Press, 2009.

[13] D. Foata and D. Zeilberger, Graphical Major Indices, *J. Comput. Appl. Math.*, **68** (1996) 79–101.

[14] O. A. Gross, Preferential arrangements, *Amer. Math. Monthly*, **69** no. 1 (1962) 4–8.

[15] M. Guan, W. Li, E. Wang, I. Gessel, T. Ferguson, C. Melolidakis, R. B. Nelsen and C. Ion, Elementary Problems: E3059-E3063, *Amer. Math. Monthly*, **91** (1984) 580–581.

[16] L. C. Hsu and P. J. Shiue, A unified approach to generalized Stirling numbers, *Adv. in Appl. Math.*, **20** no. 3 (1998) 366–384.

[17] L. Kargin and B. Cekim, Higher order generalized geometric polynomials, *Turkish J. Math.*, **42** no. 3 (2018) 887–903.

[18] E. Mendelson, Races with ties, *Math. Mag.*, **55** no. 3 (1982) 170–175.

[19] M. Mihoubi and S. Taharbouchet, Identities and congruences involving the geometric polynomials, *Miskolc Math. Notes*, **20** no. 1 (2019) 395–408.

[20] R. B. Nelsen, D. E. Knuth, J. C. Binz and G. Williams, E3062, *Amer. Math. Monthly*, **94** (1987) 376–377.

[21] R. B. Nelsen and H. Schmidt, Chains in power sets, *Math. Mag.*, **64** no. 1 (1991) 23–31.

[22] S. Nkonkobe, B. Bényi, R. B. Corcino and C. B. Corcino, A combinatorial analysis of higher order generalised geometric polynomials: A generalisation of barred preferential arrangements, *Discrete Math.*, **343** no. 3 (2020) 12 pp.

[23] S. Nkonkobe and V. Murali, A study of a family of generating functions of Nelsen Schmidt type and some identities on restricted barred preferential arrangements, *Discrete Math.*, **340** no. 5 (2017) 1122-1128.

[24] N. Pippenger, The hypercube of resistors, asymptotic expansions, and preferential arrangements, *Math. Mag.*, **83** no. 5 (2010) 331–346.

[25] N. J. A. Sloane, The on-line encyclopedia of integer sequences, `http://oeis.org`

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