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Mixed coloured permutations

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Abstract. In this paper, we introduce mixed coloured permutations, permutations with certain coloured cycles, and study the enumerative properties of these combinatorial objects. We derive the generating function, closed forms, recursions and combinatorial identities for the counting sequence, for mixed Stirling numbers of the first kind. In this comprehensive study, we consider further the conditions on the length of the cycles, r-mixed Stirling numbers and the connection to Bell polynomials.

Keywords. Stirling numbers of the first kind; r-Stirling numbers; mixed partitions; Bell polynomials.

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1. Introduction

Permutations play a unique role in combinatorics: basic objects that occur almost everywhere in different forms studied from different point of views revealing their properties in the wealth of studies. A permutation of a set S with n elements can be viewed as a function $w : \{1, 2, \ldots, n\} \to S$; $w(i) = w_i$ or as a bijection $w : S \to S$. In this latter case, for each element $x \in S$, there is a unique ℓ such that $x, w(x), \ldots, w^{\ell-1}(x)$ are different and $w^{\ell}(x) = x$. The sequence $(x, w(x,) \ldots w^{\ell-1}(x))$ is called a *cycle* of length l. Any permutation w is the unique product of distinct cycles. It is well-known that the number of permutations of $[n] = \{1, 2, \ldots, n\}$ with exactly k cycles is the signless Stirling number of the first kind, $\left[{n \atop k}\right]$. Clearly, summing up for k we obtain all permutations with a total number of $n!$. We introduce *coloured permutations* by colouring the cycles.

DEFINITION 1.1

We call a permutation π with cycle decomposition $\{C_1, C_2, \ldots, C_m\}$ and an assignment of colours from the set $\{1, 2, \ldots, k\}$ such that t_i cycles obtain the colour i, a $(t_1, t_2, \ldots, t_k)$-*coloured permutation*.

We denote the number of $(t_1, t_2, \ldots, t_k)$-coloured permutations of $[n]$ by

$$\begin{bmatrix} n \\ t_1, t_2, \ldots, t_k \end{bmatrix}.$$

Example 1.2. The following example is a (3, 1, 1)-coloured permutation of [9]:

$$(1\,6\,10)(2\,4)(3\,11\,9)(5\,7)(8) \Longleftrightarrow 6\,4\,11\,2\,7\,10\,5\,8\,3\,1\,9.$$

A permutation with k cycles can be coloured by k distinct colours in $k!$ ways; hence, the number of *distinctly coloured permutations* is

$$\begin{bmatrix} n \\ 1, 1, \ldots, 1 \end{bmatrix} = k!\begin{bmatrix} n \\ k \end{bmatrix}.$$

Though in this paper we focus on permutations, we can formulate the problem in a more general form by considering permutations of multisets instead of sets. Let $\mathcal{B} = (1^{b_1}, 2^{b_2}, \ldots, n^{b_n})$ and $\mathcal{C} = (1^{c_1}, 2^{c_2}, \ldots, k^{c_k})$ be two multisets (b_i, resp. c_i denotes the appearance of the element i in the set). Let $\left[\begin{smallmatrix} \mathcal{B} \\ \mathcal{C} \end{smallmatrix}\right]$ denote the number of permutations of the $b_1 + b_2 + \cdots + b_n$ elements of the multiset $\mathcal{B}$ into exactly $c_1 + c_2 + \cdots + c_k$ cycles such that c_j cycles are labeled by j. Further, let $\left[\begin{smallmatrix} \mathcal{B} \\ \mathcal{C} \end{smallmatrix}\right]_0$ denote the number of such permutations with at most $c_1, c_2, \ldots, c_k$ cycles. Equivalently, with $\mathcal{J} = \{1^{j_1}, 2^{j_2}, \ldots, k^{j_k}\}$, we have

$$\begin{bmatrix} \mathcal{B} \\ \mathcal{C} \end{bmatrix}_0 = \sum_{0 \le i \le k, 0 \le j_i \le c_i} \begin{bmatrix} \mathcal{B} \\ \mathcal{J} \end{bmatrix}.$$

Clearly, for $b_1 = b_2 = \cdots = b_n = 1$ and $c_1 = m$, $c_2 = c_3 = \cdots = c_k = 0$, we have

$$\begin{bmatrix} \mathcal{B} \\ \mathcal{C} \end{bmatrix} = \begin{bmatrix} n \\ m \end{bmatrix}.$$

As mentioned before, for $b_1 = b_2 = \cdots = b_n = 1$ and $c_1 = c_2 = c_3 = \cdots = c_k = 1$, we have $\left[\begin{smallmatrix} \mathcal{B} \\ \mathcal{C} \end{smallmatrix}\right] = k!\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]$. In this case, the counting sequence $\left[\begin{smallmatrix} \mathcal{B} \\ \mathcal{C} \end{smallmatrix}\right]_0$ is referred to in OEIS as A006252 [12]. Now we give a formula for the special case with $b_1 = b_2 = \cdots = b_n = 1$, but arbitrary $\mathcal{C}$. Since this special case $b_1 = b_2 = \cdots = b_n = 1$ corresponds to permutations, we write simple n instead of $\mathcal{B}$.

Theorem 1.3. *Let $n, t_1, \ldots, t_k \in \mathbb{N}$. The number of $(t_1, \ldots, t_k)$-coloured permutations is given by the following formula*:

$$\begin{bmatrix} n \\ t_1, t_2, \ldots, t_k \end{bmatrix} = \sum_{\ell_1 + \cdots + \ell_k = n} \binom{n}{\ell_1, \ell_2, \ldots, \ell_k} \begin{bmatrix} \ell_1 \\ t_1 \end{bmatrix}\begin{bmatrix} \ell_2 \\ t_2 \end{bmatrix} \cdots \begin{bmatrix} \ell_k \\ t_k \end{bmatrix}, \tag{1.1}$$

where $\binom{n}{\ell_1, \ldots, \ell_k}$ is the multinomial coefficient defined by $\frac{n!}{\ell_1! \ldots \ell_k!}$ with $\ell_1 + \cdots + \ell_k = n$.

Proof. We constitute $(t_1, t_2, \ldots, t_k)$-coloured permutations of the set $[n]$ as follows. First, choose ℓ_1 elements in $\binom{n}{\ell_1}$ ways (label them by 1) and then order them into t_1 cycles in $\left[\begin{smallmatrix} \ell_1 \\ t_1 \end{smallmatrix}\right]$ ways. Next, choose ℓ_2 out of the remaining $n - \ell_1$ elements in $\binom{n-\ell_1}{\ell_2}$ ways and order these elements into t_2 cycles in $\left[\begin{smallmatrix} \ell_1 \\ t_2 \end{smallmatrix}\right]$ ways. By continuing the process, we obtain the theorem. □

Note that $\left[{n \atop t_1,t_2,\ldots,t_k}\right]_0$ denotes the number of coloured permutations such that there are at most t_i cycles coloured by the colour i.

Theorem 1.4. *The number of* $(t_1, t_2, \ldots, t_k)$*-coloured permutations is given by*

$$\begin{bmatrix} n \\ t_1, t_2, \ldots, t_k \end{bmatrix} = \sum_{1\leqslant i\leqslant k, 0\leqslant j_i\leqslant t_i} (-1)^{\sharp(j_1,\ldots,j_k)} \begin{bmatrix} n \\ j_1, j_2, \ldots, j_k \end{bmatrix}_0,$$

where $\sharp(j_1, \ldots, j_k)$ *is the number of i's such that* $j_i \neq 0$.

Proof. The theorem follows by the inclusion–exclusion principle from the definitions. □

Theorem 1.5. *For the number of* $(t_1, t_2, \ldots, t_k)$*-coloured permutations, the following recurrence holds:*

$$\begin{aligned} \begin{bmatrix} n \\ t_1, t_2, \ldots, t_k \end{bmatrix} &= (n-1) \begin{bmatrix} n-1 \\ t_1, t_2, \ldots, t_k \end{bmatrix} \\ &\quad + \sum_{j=1}^{k} \begin{bmatrix} n-1 \\ t_1, \ldots, t_{j-1}, t_j - 1, t_{j+1} \ldots, t_k \end{bmatrix}. \end{aligned}$$

Proof. Consider the n-th element. If it is a singleton coloured by the colour j, the remaining elements construct a $(t_1, \ldots, t_{j-1}, t_j - 1, t_{j+1}, \ldots, t_k)$-coloured permutation. If n is not a singleton, we can insert it before any of the elements and join it to the cycle of this element which gives $(n-1)\left[{n-1 \atop t_1,t_2,\ldots,t_k}\right]$ possibilities. □

In the rest of the paper, we consider a special case.

DEFINITION 1.6

We call a $(t, 1, \ldots, 1)$-coloured permutation (with $k-1$ 1's; notice that k denotes the number of different colours used in the permutation) *mixed coloured permutation*. We denote the set of mixed coloured permutations with $t + k - 1$ cycles by $\mathcal{MC}(n, k, t)$ and denote the size of this set by $\left[{n \atop t,1,\ldots,1}\right] = \left[{n \atop k/t}\right]$. We call the number sequence $\left[{n \atop k/t}\right]$ *mixed Stirling number of the first kind*. We refer to the colour that is used for t cycles as the *special colour*.

Table 1 lists the number of mixed coloured permutations for some small values of n, k and t.

For the special case $t = 1$, when every cycle is distinctly coloured, we have the following recurrence relation.

Theorem 1.7. *For positive integers* n, k *with* $k \leq n$, $\left[{n \atop k/1}\right]$ *satisfies*

$$\begin{bmatrix} n \\ k/1 \end{bmatrix} = k \begin{bmatrix} n-1 \\ k-1/1 \end{bmatrix} + (n-1) \begin{bmatrix} n-1 \\ k/1 \end{bmatrix}. \tag{1.2}$$

Table 1. $\left[{n \atop k/2}\right]$ and $\left[{n \atop k/3}\right]$.

n/k	1	2	3	4	5	n/k	1	2	3	4	5
2	1					3	1				
3	3	3				4	6	4			
4	11	18	12			5	35	40	20		
5	50	105	120	60		6	225	340	300	120	
6	274	675	1020	900	360	7	1624	2940	3500	2520	840

Proof. Consider the n-th element. If it is a fixed point, then there are $\left[{n-1 \atop k-1/1}\right]$ ways to create the mixed coloured permutation from the remaining elements. The cycle with n can be coloured by any of the k colours. If it is not a fixed point, proceed as follows: write the permutation in cycle notation (the elements in a cycle are arranged so that the least element is written first). This can be done in $\left[{n-1 \atop k/1}\right]$ ways. Insert now the element n before any of the elements. If we insert n before an element which started a cycle, n will be included into this cycle. This gives $(n-1)\left[{n-1 \atop k/1}\right]$ possibilities. □

Theorem 1.8. *Let n, k and t be positive integers with $t, k \leq n$. Then the number of mixed coloured permutations is*

$$\left[{n \atop k/t}\right] = \sum_{\ell=t}^{n-k+1} (k-1)!\binom{n}{\ell}\left[{\ell \atop t}\right]\left[{n-\ell \atop k-1/1}\right].$$

Proof. Choose $\ell \geqslant t$ elements in $\binom{n}{\ell}$ ways and order them into t cycles. These cycles are coloured by the special colour. The remaining $n-\ell$ elements have to be ordered into $k-1$ distinctly coloured cycles. This can be done in $\left[{n-\ell \atop k-1/1}\right]$ ways. Note that we should have $k-1 \leqslant n-\ell$. □

COROLLARY 1.9

Let n, k and t be positive integers with $k, t \leq n$. Then we have

(*i*) $\left[{n \atop k/0}\right] = \left[{n \atop k-1/1}\right]$,
(*ii*) $\left[{n \atop 1/t}\right] = \left[{n \atop t}\right]$,
(*iii*) $\left[{n \atop 1/n-1}\right] = \binom{n}{2} = \left[{n \atop n-1}\right]$,
(*iv*) $\left[{n \atop 2/n-1}\right] = n$,
(*v*) $\left[{n \atop n-t+1/t}\right] = \frac{n!}{t!}$.

We derive some formulas using combinatorial arguments. We use the notation of the falling factorial $(n)_k = n(n-1)\cdots(n-k+1)$.

Theorem 1.10. *For positive integers n, k and t with $k, t \leq n$, the number of mixed coloured permutations is*

$$\begin{bmatrix} n \\ k/t \end{bmatrix} = (t+k-1)_{k-1} \begin{bmatrix} n \\ t+k-1 \end{bmatrix}. \tag{1.3}$$

Proof. First, we arrange the n elements into $t+k-1$ cycles, then we choose t cycles to colour them with the special colour and colour the remaining $k-1$ cycles with distinct colours. □

We present a few more expressions to calculate the number of mixed coloured permutations.

Theorem 1.11. *For positive integers n, k and t with $k, t \leq n$, we have*

$$\begin{bmatrix} n \\ k/t \end{bmatrix} = \sum_{j=t}^{n-k+1} (k-1)! \binom{n}{j} \begin{bmatrix} j \\ t \end{bmatrix} \begin{bmatrix} n-j \\ k-1 \end{bmatrix}. \tag{1.4}$$

Proof. We first choose j elements and create from these elements t cycles in $\binom{n}{j}\left[\begin{smallmatrix} j \\ t \end{smallmatrix}\right]$ ways. The remaining $n-j$ elements are included in the $k-1$ cycles and since these are distinctly coloured, we have a factor $(k-1)!\left[\begin{smallmatrix} n-j \\ k-1 \end{smallmatrix}\right]$. □

Theorem 1.12. *Let n, k and t be positive integers with $k, t \leq n$. Then*

$$\begin{bmatrix} n \\ k/t \end{bmatrix} = \begin{bmatrix} n-1 \\ k/t-1 \end{bmatrix} + (k-1)\begin{bmatrix} n-1 \\ k-1/t \end{bmatrix} + (n-1)\begin{bmatrix} n-1 \\ k/t \end{bmatrix}.$$

Proof. Consider the n-th element. If it is a singleton, it can be coloured by the special colour or by any of the other $(k-1)$ colours. These are $\left[\begin{smallmatrix} n-1 \\ k/t-1 \end{smallmatrix}\right] + (k-1)\left[\begin{smallmatrix} n-1 \\ k-1/t \end{smallmatrix}\right]$ possibilities. If it is not a singleton, it can be inserted before any element and added to the cycle of the element before it was inserted, which can be done in $(n-1)\left[\begin{smallmatrix} n-1 \\ k/t \end{smallmatrix}\right]$ ways. □

Theorem 1.13. *For positive integers n, k and t with $k, t \geq n$, we have*

$$\begin{bmatrix} n \\ k/t \end{bmatrix} = \sum_{j=1}^{n} \binom{n}{j} (j-1)! \begin{bmatrix} n-j \\ k-1/t \end{bmatrix}. \tag{1.5}$$

Proof. Mark one cycle that is not coloured by the special colour. This can be done in $(k-1)\left[\begin{smallmatrix} n \\ k/t \end{smallmatrix}\right]$ ways. Otherwise, we can build a cycle of length j in $\binom{n}{j}(j-1)!$ ways which we colour in any of the $(k-1)$ non-special colours and arrange the remaining elements into a mixed coloured permutation in $\left[\begin{smallmatrix} n-j \\ k-1/t \end{smallmatrix}\right]$ ways. After simplification by the factor $(k-1)$, we obtain the theorem. □

Next, we derive the generating functions for the enumerations of the sets $\mathcal{MC}(n, k, t)$ using the symbolic method [8]. The theory states that the generating function of a set of combinatorial objects can be directly obtained according to a symbolic construction built up of classical basic constructions as sets (SET), sequences (SEQ), cycles (CYC), etc. We recall briefly the results from [8] that we use here in order to facilitate to follow our proof for readers less familiar with the symbolic method.

Let $\mathcal{A}$ and $\mathcal{B}$ be combinatorial classes with exponential generating functions

$$A(x) = \sum_{\alpha\in\mathcal{A}} \frac{x^{|\alpha|}}{|\alpha|!} = \sum_{n\geq 0} a_n \frac{x^n}{n!} \quad \text{and} \quad B(x) = \sum_{\beta\in\mathcal{B}} \frac{x^{|\beta|}}{|\beta|!} = \sum_{n\geq 0} b_n \frac{x^n}{n!}.$$

a_n (resp. b_n) is the counting sequence for objects in the class $\mathcal{A}$ (resp. $\mathcal{B}$) with size n. Let $\mathcal{X}$ be the atomic class with generating function x. Here we need the following constructions beyond the sum and the product:

- $\mathrm{SEQ}_k(\mathcal{A})$ stands for the class of k-sequences. A k-sequence is a sequence of length k with parts in $\mathcal{A}$. The translation rule is $(A(x))^k$. $\mathrm{SEQ}(\mathcal{A})$ denotes the class of sequences without taking the length of the sequence into account. The translation rule is $\frac{1}{1-A(x)}$.
- $\mathrm{SET}_k(\mathcal{A})$ denotes the class of k-sets formed from $\mathcal{A}$, a k-sequence modulo the equivalence relation that two sequences are equivalent when the components of one of them is the permutation of the components of the other. The corresponding rule is $\frac{(A(x))^k}{k!}$. $\mathrm{SET}(\mathcal{A})$ is the class of sets with the translation rule $\exp(A(x))$.
- The notation $\mathrm{CYC}_k(\mathcal{A})$ is used for k-cycles, k-sequences modulo the equivalence relation identifying sequences whose elements are cyclic permutations of each other. The rule is $\frac{(A(x))^k}{k}$. $\mathrm{CYC}(\mathcal{A})$ is the class of all cycles and corresponds to $\log\frac{1}{1-A(x)}$.

Theorem 1.14. *The exponential generating function of the mixed Stirling number of the first kind is given by*

$$\sum_{n=0}^{\infty} \begin{bmatrix} n \\ k/t \end{bmatrix} \frac{x^n}{n!} = \frac{1}{t!}\left(\log\frac{1}{1-x}\right)^{t+k-1}. \tag{1.6}$$

Proof. We need to know how we can get a mixed coloured permutation using the basic constructions listed before. A mixed coloured permutation is actually cycles with two extra structures: a set with t elements and an arrangement with $k-1$ elements. We could say it is a pair of t-set of cycles and of $(k-1)$-sequence of cycles. Hence, with little abuse of notation, the construction for the mixed coloured permutations is the following:

$$\mathcal{MC}(n,k,t) = \mathrm{SET}_t(\mathrm{CYC}(\mathcal{X})) \times \mathrm{SEQ}_{k-1}(\mathrm{CYC}(\mathcal{X})). \tag{1.7}$$

The translation rule gives

$$\frac{\left(\log\frac{1}{1-x}\right)^t}{t!} \times \left(\log\frac{1}{1-x}\right)^{k-1}$$

which implies (1.6) after simplification. □

2. S-restricted Stirling number of the first kind

In a series of papers about Stirling numbers [2–4], the authors studied the underlying objects, partitions, permutations and lists, with the extra condition on the size of the sets, cycles and lists, respectively. For the sake of a comprehensive study of mixed Stirling numbers of the first kind, we follow this idea and derive some results for the number of mixed permutations such that each cycle has length s contained in a given set of integers S. These general results include many interesting cases that can be easily obtained by special settings of S, as for instance, $S = \{1, 2, \dots, m\}$ the restricted mixed Stirling number of the first kind and $S = \{m, m+1, \dots\}$ the associated mixed Stirling number of the first kind. Furthermore, we can set S as the set of even numbers or the set of odd numbers. Moreover, by an appropriate choice of S, our results include results for mixed permutations with forbidden cycle lengths.

DEFINITION 2.1

Given a set S of positive integers and n, k, t positive integers with $k, t \leq n$, we let $\mathcal{MC}_S(n, k, t)$ denote the set of permutations of $\{1, 2, \dots, n\}$ into k cycles such that

(a) each cycle has size given in S, and
(b) t cycles are coloured with the special colour and the remaining $k-1$ with distinct colours.

We let $\left[{n \atop k/t}\right]_S$ denote the size of the set $\mathcal{MC}_S(n, k, t)$.

First, we recall the definition, generating function and a recurrence of the S-restricted Stirling numbers of the first kind on which we rely upon in this section. For a given set of positive integers S and S-restricted Stirling numbers, $\left[{n \atop k}\right]_S$ enumerates the set of permutations of an n element set into k cycles such that each cycle has size contained in S. In other words, we get permutations with cycle index $(c_1, c_2, \dots, c_n)$ such that $c_i \neq 0$ if and only if $i \in S$. This number array is a special case of the (S, r)-Stirling numbers of the first kind $\left[{n \atop k}\right]_{r,S}$, defined in [3]. The generating function and the basic recursion are as follows:

$$\sum_{n=k}^{\infty} \left[{n \atop k}\right]_S \frac{x^n}{n!} = \frac{1}{k!}\left(\sum_{i\geq 1} \frac{x^{k_i}}{k_i}\right)^k,$$

$$\left[{n+1 \atop k}\right]_S = \sum_{s\in S}(s-1)!\binom{n}{s-1}\left[{n-s+1 \atop k-1}\right]_S.$$

Theorem 2.2. *Given a set of integers S and positive integers n, k and t with $k, t < n$, we have*

$$\left[{n \atop k/t}\right]_S = (t+k-1)_{k-1}\left[{n \atop t+k-1}\right]_S,$$

$$\left[{n \atop k/t}\right]_S = \sum_{j=t}^{n-k+1}(k-1)!\binom{n}{j}\left[{j \atop t}\right]_S\left[{n-j \atop k-1}\right]_S,$$

$$\left[{n \atop k/t}\right]_S = \sum_{s\in S}\binom{n}{s}(s-1)!\left[{n-s \atop k-1/t}\right]_S.$$

Proof. The identities are straightforward consequences of the equations (1.3), (1.4) and (1.5). □

Theorem 2.3. *For a given set of positive integers S, the exponential generating function for $\left[{n \atop k/t}\right]_S$ is given by the formula*

$$\sum_{n=0}^{\infty}\left[{n \atop k/t}\right]_S \frac{x^n}{n!} = \frac{1}{t!}\left(\sum_{s\in S}\frac{x^s}{s}\right)^{t+k-1}. \tag{2.1}$$

Proof. The proof is analogous to the proof of (1.6). We only need to ensure the condition on the length of the cycles. We use the symbols CYC_S for cycles such that the length of the cycle is included in a given fixed set S. Hence, in this case the construction (1.7) has to be modified to

$$\mathcal{MC}_S(n,k,t) = \mathrm{SET}_t(\mathrm{CYC}_S(\mathcal{X})) \times \mathrm{SEQ}_{k-1}(\mathrm{CYC}_S(\mathcal{X})),$$

which translates to

$$\frac{1}{t!}\left(\sum_{s\in S}\frac{x^s}{s}\right)^{t} \times \left(\sum_{s\in S}\frac{x^s}{s}\right)^{k-1}.$$

□

Next, we derive some formulas for specific S sets.

Theorem 2.4. *For a given set of positive integers S with $1 \in S$, we have*

$$\left[{n \atop k/t}\right]_S = \sum_{i=0}^{t}\sum_{j=0}^{k-1}\binom{n}{i,\, j}(k-1)_j\left[{n-i-j \atop k-j/t-i}\right]_{S-\{1\}}.$$

Proof. Let i be the number of fixed points coloured with the special colour and j the number of fixed points coloured with any of the other colours. Choose the i elements out of n elements. Choose j elements out of $n-i$ elements and colour them with one of the $(k-1)$ colours in $(k-1)(k-2)\cdots(k-j+1)$ ways. The remaining $n-i-j$ elements are ordered into a mixed coloured permutations without fixed points. □

Let $\left[{n \atop k/t}\right]_{>1}$ denote the *mixed coloured derangements*, i.e., permutations without fixed points.

COROLLARY 2.5

For positive integers n, k and t with $k, t < n$, we have

$$\left[{n \atop k/t}\right] = \sum_{i=0}^{t}\sum_{j=0}^{k-1}\binom{n}{i,\, j}(k-1)_j\left[{n-i-j \atop k-j/t-i}\right]_{>1}.$$

This theorem can be formulated in a more general form, not only for fixed points, i.e., cycles for length 1, but for any cycles with fixed length u. $S - \{u\}$ denotes the set S excluded from the element u.

Theorem 2.6. *If $u \in S$, the following identity holds*:

$$\begin{bmatrix} n \\ k/t \end{bmatrix}_S = \sum_{i=0}^{t} \sum_{j=0}^{k-1} \binom{n}{u(i+j)} \frac{(ui+uj)!}{u^{i+j} i! j!} (k-1)_j \begin{bmatrix} n-u(i+j) \\ k-j/t-i \end{bmatrix}_{S-\{u\}}.$$

Proof. Let i be the number of cycles of length u coloured by the special colour and j the number of cycles of length u coloured by any other colour. We choose $u(i+j)$ elements and order them into cycles of length u according to the following procedure: we arrange the elements and take the first u elements as a cycle, then the next u elements as a next cycle and so on. Clearly, some double counts occur: any of the elements in a cycle could be the starting element and the order of the cycles are not important. Only the fact is crucial that the first i will be coloured by the special colour. Taking these aspects into account, we find that there are $\binom{n}{u(i+j)} \frac{(u(i+j))!}{u^{i+j} i! j!}$ possibilities. We need to choose the colours for the non-special coloured cycles in $(k-1)_j$ ways. The remaining $n - u(i+j)$ elements build a mixed permutation without any cycle of length u. □

3. Mixed r-Stirling numbers of the first kind

Recently, the r-Stirling numbers have received a lot of attention. The r-Stirling numbers of the first kind counts the number of permutations that can be decomposed into exactly k cycles such that the first r elements $\{1, 2, \ldots, r\}$ are in distinct cycles. In this section, we introduce r-mixed Stirling numbers of the first kind following the original concept.

DEFINITION 3.1

We call a mixed coloured permutation *r-mixed coloured permutation* if the elements $\{1, 2, \ldots, r\}$ are in distinct cycles. We let $\mathcal{MC}_r(n, k, t)$ denote the set of all r-mixed $(t, 1, \ldots, 1)$-coloured permutations of the set $\{1, 2, \ldots, n\}$ coloured by exactly k colours. We let $\begin{bmatrix} n \\ k/t \end{bmatrix}_r$ denote the size of the set $\mathcal{MC}_r(n, k, t)$.

Theorem 3.2. *Let n, k, t and r be positive integers. The number of r-mixed coloured permutations is*

$$\begin{bmatrix} n \\ k/t \end{bmatrix}_r = (t+k-1)_{k-1} \begin{bmatrix} n \\ t+k-1 \end{bmatrix}_r.$$

Proof. First, order the n elements into $t+k-1$ cycles such that the elements $\{1, 2, \ldots, r\}$ are in distinct cycles. Next, colour the cycles to obtain a mixed coloured permutations, choose $k-1$ cycles for the $k-1$ colours. □

The next formula contains the *Lah numbers*, $\left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rfloor$, which count the number of partitions of an n element set into k linearly ordered subsets and is given by the closed formula [9]. One of the known closed formula for Lah numbers is [9]

$$\left\lfloor {n \atop k} \right\rfloor = \binom{n-1}{k-1}\frac{n!}{k!}.$$

Theorem 3.3. *Let n, k, t and r be positive integers. The r-mixed Stirling numbers of the first kind is given by*

$$\left[{n \atop k/t}\right]_r = \sum_{\ell=k+t-r-1}^{n-r}\sum_{i=0}^{\min(t,r)} r!\binom{r}{i}\binom{k-1}{r-i}\binom{n-r}{\ell} \left\lfloor {n-r-\ell \atop r} \right\rfloor \left[{\ell \atop k-r+i/t-i}\right].$$

Proof. First, put the r elements $\{1, 2, \ldots, r\}$ into distinct cycles and colour these cycles. Let i be the number of cycles that contain one of the r elements from the set $\{1, 2, \ldots, r\}$ and are coloured by the special colour $(0 \le i \le \min(t, r))$. There are $\sum_i \binom{r}{i}\binom{k-1}{r-i}(r-i)!$ possibilities to do so by choosing the i elements for the special colour, choosing the colours, and order the colours for the remaining $r - i$ cycles. Next, we construct a mixed coloured permutation without any elements of the set $\{1, 2, \ldots, r\}$. Choose ℓ elements for this out of the set $\{r+1, r+2, \ldots, n\}$ in $\left[{\ell \atop k-(r-i)/t-i}\right]$ ways. Finally, construct the r cycles containing the elements $\{1, 2, \ldots, r\}$ with the remaining $n - r - \ell$ elements by partitioning them into r linearly ordered sets and assigning to each list a number between 1 and r. $\square$

4. Partial Bell polynomials and mixed permutations

We modify partial Bell polynomials so that the obtained polynomials gives at certain values mixed Stirling numbers of the first kind. Further, we will see that these polynomials connect the theory of mixed coloured permutations and mixed partitions [1,13]. So this variation of Bell polynomials is an unified approach to both the theory and could be easily extended, for instance, to the theory of mixed coloured lists or mixed Lah numbers generalizing the well-known Lah numbers in the same vein.

Bell polynomials, $B_{n,k}(x_1, x_2, \ldots) = B_{n,k}(x_j)_{j\ge1}$ were introduced by Bell as a mathematical tool for representing the n-th derivative of composite functions and were intensively studied since then by many authors [7,10]. The exponential partial Bell polynomial is defined by the generating function

$$\sum_{n\ge k} B_{n,k}(x_j)\frac{t^n}{n!} = \frac{1}{k!}\left(\sum_{m\ge1} x_m \frac{t^m}{m!}\right)^k \tag{4.1}$$

and is given explicitly by

$$B_{n,k}(x_1, x_2, \ldots) = \sum_{\pi(n,k)} \frac{n!}{k_1!\cdots k_n!}\left(\frac{x_1}{1!}\right)^{k_1}\left(\frac{x_2}{2!}\right)^{k_2}\cdots\left(\frac{x_n}{n!}\right)^{k_n}, \tag{4.2}$$

where $\pi(n, k) = \{(k_1, \ldots, k_n) \in N^n : k_1 + k_2 + \cdots + k_n = k, k_1 + 2k_2 + \cdots nk_n = n\}$. For instance, $B_{6,2}(x_2, x_3, x_4) = 15x_2x_4 + 10x_3$ since $\{1, 2, \ldots, 6\}$ can be partitioned into two blocks with size 2/4 and 3/3 in 15 and 10 ways, respectively. It is well-known [7] that for particular settings of the variables x_j, the exponential partial Bell polynomials reduce to the following sequences:

$$B_{n,k}(1, 1, 1, \ldots) = \left\{ {n \atop k} \right\},$$
$$B_{n,k}(0!, 1!, 2!, \ldots) = \left[{n \atop k} \right],$$
$$B_{n,k}(1!, 2!, 3!, \ldots) = \left\lfloor {n \atop k} \right\rfloor.$$

So Bell polynomials can be viewed as a unified approach to all three problems: partitions, cycles and lists of sets.

Mihoubi *et al.* [10] generalized this type of Bell polynomials for a general sequence $(a_j)_{j\geq 1}$ and presented the following combinatorial interpretation: For a given $(a_j)_{j\geq 1}$ sequence, $B_{n,k}(a_j)$ counts the number of partitions of an n element set into k blocks such that the blocks of size j can be coloured by a_j colours, $B_{n,k}((j-1)!a_j)$ counts the number of permutations of an n element set into k cycles such that the cycles of size j can be coloured by a_j colours, and $B_{n,k}(j!a_j)$ counts the number of partitions of an n element set into k lists such that the lists of size j can be coloured by a_j colours.

Moreover, the authors extend this point of view for the cases of r-Stirling and r-Lah numbers by defining the r-partial Bell polynomials $B_{n,k}^{(r)}(x_i, y_i)$. Then

$$B_{n,k}^{(r)}(x_i, y_i) = \sum_{p(n,k,r)} \left[\frac{n!}{k_1!k_2!\cdots} \left(\frac{x_1}{1!}\right)^{k_1} \left(\frac{x_2}{2!}\right)^{k_2} \cdots \right] \left[\frac{r!}{r_0!r_1!\cdots} \left(\frac{y_1}{1!}\right)^{k_1} \left(\frac{y_2}{2!}\right)^{k_2} \cdots \right], \tag{4.3}$$

where $p(n, k, r)$ is the set of all nonnegative integer sequences $(k_i)_{i\geq 1}$ and $(r_i)_{i\geq 0}$ such that $\sum_{i\geq 1} k_i = k$, $\sum_{i\geq 0} r_i = r$ and $\sum_{i\geq 1} i(k_i + r_i) = n$.

Clearly, $B_{n,k}^{(0)}(x_i, y_i) = B_{n,k}(x_i)$. Further, the special cases reduce to the well-known sequences:

$$B_{n,k}^{(r)}(1, 1, 1 \ldots) = \left\{ {n \atop k} \right\}_r \quad (r\text{-Stirling numbers of the second kind [5]}),$$
$$B_{n,k}^{(r)}(0!, 1!, 2!, \ldots) = \left[{n \atop k} \right]_r \quad (r\text{-Stirling numbers of the first kind [5]}),$$
$$B_{n,k}^{(r)}(1!, 2!, 3!, \ldots) = \left\lfloor {n \atop k} \right\rfloor_r \quad (r\text{-Lah numbers [11]}),$$
$$B_{n,k}^{(r)}(1, m, m^2, \ldots; 1, 1, \ldots) = W_{m,r}(n, k) \quad (r\text{-Whitney numbers [6]}).$$

In order to derive a similar unified approach to the mixed Stirling numbers (and mixed Lah numbers, though these numbers are not introduced yet), we modify the above definition slightly.

DEFINITION 4.1

Let n, k and t be integers and let $(x_\ell)_{\ell\geq 1}$ and $(y_\ell)_{\ell\geq 1}$ be the sequences of the integers

$$B^*_{n,k,t}(x_\ell; y_\ell) = \sum_{p^*(n,k,t)} \frac{n!}{t_1!t_2!\cdots t_n!}\left(\frac{x_1}{1!}\right)^{t_1}\left(\frac{x_2}{2!}\right)^{t_2}\cdots\left(\frac{x_n}{n!}\right)^{t_n}$$
$$\frac{(k-1)!}{k_1!k_2!\cdots k_n!}\left(\frac{y_1}{1!}\right)^{k_1!}\left(\frac{y_2}{2!}\right)^{k_2}\cdots\left(\frac{y_n}{n!}\right)^{k_n},$$

where the sum runs over all sequences $p^*(n,k,t) = \{(t_i)_{i\geq 1}; (k_i)_{i\geq 1} : k_i, t_i \in N, \sum_{i\geq 1} t_i = t, \sum_{i\geq 1} k_i = k-1, \sum_{i\geq 1} i(t_i + k_i) = n\}$.

It is well-known that the number of partitions of $[n]$ into k blocks is given by the following formula, where k_i denotes the number of blocks with cardinality i:

$$\sum_{\pi(n,k)} \frac{n!}{k_1!(1!)^{k_1}k_2!(2!)^{k_2}\cdots k_n!(n!)^{k_n}},$$

where $\pi(n,k) = \{(k_1,\ldots,k_n) \in N^n : k_1 + k_2 + \cdots + k_n = k, k_1 + 2k_2 + \cdots nk_n = n\}$.

From this fact and the previous results, we recall that the partial and r-partial Bell polynomials of the following combinatorial interpretation should be straightforward.

Theorem 4.2. $B^*_{n,k,t}(x_1, x_2, \ldots; y_1, y_2, \ldots)$ *counts the number of partitions of* n *into* t *blocks such that the blocks of length* i *can be coloured with* x_i *colours and* $k-1$ *ordered blocks such that a block of length* i *can be coloured with* y_i *colours.*

We consider explicitly the case when $x_i = y_i$, the case when we do not have to distinguish between the two types of sequences. First, we define Bell polynomials associated to such a sequence combinatorially and then we present some formulas for this case.

DEFINITION 4.3

Let $(a_n; n \geq 1)$ be a sequence of non-negative integers. The number $B^*_{n,k,t}(a_j)$ $(x_i = y_i = a_i)$ counts the number of mixed partitions of an n element set into k blocks such that t blocks are labeled by 1 and the other blocks by $2,\ldots,k$ such that any block of size j can be coloured with a_j colours.

Theorem 4.4. *For* n, k *and* t *positive integers with* $k, t \leq n$ *and a given vector of positive integers* $(a_1, a_2, \ldots)$, *the following statements hold:*

$$B^*_{n,k,t}(a_1, a_2, \ldots) = \frac{1}{t!} \sum_{\sum_{i=1}^{n+k-t} n_i = n} \binom{n}{n_1, n_2, \ldots, n_{k+t-1}} a_{n_1}a_{n_2}\cdots a_{n_{k+t-1}}, \tag{4.4}$$

$$B^*_{n,k,t}(a_j) = \frac{n!}{t!} \sum_{\sum_{i=1}^{k+r-1} n_i = n} \frac{a_{n_1}a_{n_2}\cdots a_{n_{k+t-1}}}{n_1!n_2!\cdots n_{k+t-1}!}, \tag{4.5}$$

$$\sum_{n=k}^{\infty} B^*_{n,k,t}(a_j)\frac{z^n}{n!} = \frac{1}{t!}\left(\sum_{m\geq 1} a_m \frac{z^m}{m!}\right)^{t+k-1}. \tag{4.6}$$

Proof. There are $\frac{1}{t!}\binom{n}{n_1,n_2,\ldots,n_t}a_{n_1}a_{n_2}\cdots a_{n_t}$ possibilities to partition elements out of the set $\{1, 2, \ldots, n\}$ into t blocks of sizes $n_1, n_2, \ldots, n_t$, and colour them by $a_{n_1}, a_{n_2}, \ldots, a_{n_t}$ colours. The remaining elements $n - (n_1 + n_2 \ldots + n_t)$ can be arranged into an ordered partition with blocks of sizes n_{t+1}, n_{t+2}, …, n_{t+k-1} and can be coloured with $a_{n_{t+1}}$, $a_{n_{t+2}}, \ldots, a_{n_{t+k-1}}$ colours in $\binom{n-(n_1+n_2\ldots+n_t)}{n_{t+1},n_{t+2},\ldots,n_{t+k-1}}$ ways. This implies (4.4). We get (4.5) from (4.4) by simplification. We present the derivation of equation (4.6) explicitly:

$$\begin{aligned}\sum_{n\geq \max k,t} B^*_{n,k,t}(a_1, a_2, \ldots)\frac{z^n}{n!} &= \sum_{n\geq \max k,t}\frac{1}{t!}\sum_{n_1+\cdots+n_{t+k-1}=n}\frac{a_1a_2\cdots a_{t+k-1}}{n_1!n_2!\cdots n_{t+k-1}!}z^n\\ &= \frac{1}{t!}\sum_{n_i\geq 1}\frac{a_{n_1}\cdots a_{n_{t+k-1}}}{n_1!\cdots n_{t+k-1}!}z^{n_1+\cdots n_{t+k-1}}\\ &= \frac{1}{t!}\left(\sum_{j\geq 1} a_j\frac{z^j}{j!}\right)^{t+k-1},\end{aligned}$$

and hence (4.6) follows. □

We are now ready to connect the theory of our newly introduced numbers, the mixed Stirling numbers of the first kind to the theory of Bell polynomials. Clearly, we have as follows.

Theorem 4.5. *For positive integers n, k and t with $t, k \leq n$, we have*

$$\begin{bmatrix} n \\ k/t \end{bmatrix} = B^*_{n,k,t}(0!, 1!, 2!, \ldots).$$

Further, it is also easy to give a formula using $B^*(n, k, t)(a_i)$ for the S-restricted case $\begin{bmatrix} n \\ k/t \end{bmatrix}_S$. In order to keep the form of the statement nice, we introduce the following notation.

Let c_S be the *characteristic sequence* of a given set S of positive integers, i.e., $c_S = \{c_i | i = 1, 2, \ldots, c_i = 1 \text{ if } i \in S \text{ and } c_i = 0 \text{ if } i \notin S\}$.

Theorem 4.6. *Given n, k, t positive integers with $k, t \leq n$ and a set of positive integers S, we have*

$$\begin{bmatrix} n \\ k/t \end{bmatrix}_S = B^*_{n,k,t}((i-1)!c_S).$$

Furthermore, based on Mihoubi [10], we can interpret the general expression $B^*_{n,k,t}((i-1)!a_i)$ combinatorially.

Theorem 4.7. *Given n, k, t positive integers with $k, t \leq n$ and a sequence of integers $(a_i)_i \geq 1$, $B^*_{n,k,t}((i-1)!a_i)$ is the number of permutations of $[n]$ into $k+t-1$ double labeled cycles: the first label of t cycles is* 1, *while the first labels of the other cycles are different out of the set $\{2, 3, \ldots, k\}$. The second label depends on the size of the cycle, each cycle of length i receives a label out of the set $\{1, 2, \ldots, a_i\}$.*

The detailed study of the Bell polynomials $B^*_{n,k,t}(x_i, y_i)$ is beyond the goal of this paper, but it is important to mention the connections to mixed partitions and to mixed Stirling numbers of the second kind whose objects were studied by Barati *et al.* [1] and Yaqubi *et al.* [13].

In particular, $B^*_{n,k,t}(1, 1, \ldots)$ counts the number of mixed partitions defined and studied in [1,13]. Further, $B^*_{n,k,t}(c_S)$ gives the number of mixed partitions with block sizes contained in a given set of positive integers.

Similar statements could be formulated about mixed lists and mixed Lah numbers.

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References

[1] Barati S, Bényi B, Jafarzaded A and Yaqubi D, Mixed restricted Stirling numbers, *Acta Math. Hungar.* (2019) https://doi.org/10.1007/s10474-019-00915-8

[2] Bényi B and Ramírez J L, Some applications of S-restricted set partitions, *Period. Math. Hung.* **78(1)** (2019) 110–127

[3] Bényi B, Méndez M, Ramírez J L and Wakhare T, Restricted r-Stirling numbers and their combinatorial applications, *Appl. Math. Comput.* **348** (2019) 186–205

[4] Bényi B, Méndez M and Ramírez J L, Generalized ordered set partitions, preprint (2019)

[5] Broder A Z, The r-Stirling numbers, *Discrete Math.* **49** (1984) 241–259

[6] Cheon G S and Jung J H, r-Whitney numbers of Dowling lattices, *Discrete Math.* **312** (2012) 2337–2348

[7] Comtet L, Advanced combinatorics, (1974) (Dordrecht, Holland: D. Reidel Publishing Co.)

[8] Flajolet P and Sedgewick R, Analytic combinatorics (2009) (Cambridge: Cambridge University Press)

[9] Lah I, A new kind of numbers and its application in the actuarial mathematics, *Boletin do Instituto. dos Actuarios Portugueses* **9** (1954) 7–15

[10] Mihoubi M and Rahmani M, The partial r-Bell polynomials, *Afrika Mat.* **28(7-8)** (2017) 1167–1183

[11] Nyul G and Rácz G, The r-Lah numbers, *Discrete Math.* **338** (2015) 1660–1666

[12] Sloane N J A, The On-Line Encyclopedia of Integer Sequences, URL http://oeis.org

[13] Yaqubi D, Mirzavaziri M and Saeednezhad Y, Mixed r-Stirling number of the second kind, *Online J. Anal. Comb.* **11** (2016) 5

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