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CYCLOCOPULA TECHNIQUE TO STUDY THE RELATIONSHIP BETWEEN TWO CYCLOSTATIONARY TIME SERIES WITH FRACTIONAL BROWNIAN MOTION ERRORS

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Abstract

Detection of relationship between two time series is so important in different scientific fields. Most common techniques are usually sensitive to stationarity or normality assumptions. In this research, a new copula-based method (cyclocopula) is introduced to detect the relationship between two cylostationary time series with fractional Brownian motion (fBm) errors. The performance of the proposed method is studied by employing numerous simulated datasets. The applicability of the introduced approach is also investigated in real-world problems. The numerical and applied studies verify the performance of the introduced technique.

Keywords: Time Series; Fractional Brownian Motion; Cyclostationary; Copula; Regression; Time Series Analysis; Time Series Analysis; Time Series Forecasting.

1. INTRODUCTION

The ways of modeling dependency between two time series have always been one of the main focuses in practice. The choice of a better model depends on the dependency structure of obligators and is a crucial part of modeling. Pearson's correlation coefficient (or equivalently, simple linear regression),[1–6] Spearman's correlation coefficient,[7–10] Kendall's correlation coefficient,[11–14] Sen's slope,[13,15,16] cross-correlation function[1,17,18] and copula[19–24] are some common methods that have been frequently applied to investigate the relationship of two stationary time series. Cross-correlation function, similar to linear regression, is somewhat sensitive to the existence of outliers and abnormality of datasets. An efficient alternative method to investigate the dependency of two abnormal (or outliers-equipped) stationary time series is to use copula. Copula is a function which transfers the multivariate distribution function to its marginal distribution function by quantiles. Copula helps understand the correlation beyond linearity and is a great tool for modeling dependency where correlation follows a random distribution. Copula technique is most efficient for stationary time series and may not work well for nonstationary time series such as the cyclostationary (CS) time series. In this research, for the first time, the relationship between two cyclostationary time series with fractional Brownian motion (fBm) errors is considered. To solve this issue, we introduce a copula-based regression approach, called the cyclocopula technique. The ability of the proposed approach to detect relationship between two cyclostationary time series with fBm errors is studied by employing numerous datasets. The applicability of the introduced approach is also investigated in real-world problems.

2. METHODOLOGY

2.1. Copula

Copula was first introduced by Sklar[25] as a statistical mechanism to transfer the joint distribution into its marginals and as a model to show the dependency between the marginals. Copulas are functions that link marginal distributions to the multivariate distributions.[26] Consider two continuous random variables X and Y with the distribution functions

$$F(x) = \Pr(X \leq x)$$

and

$$G(y) = \Pr(Y \leq y),$$

respectively, and with the joint distribution function

$$H(x, y) = \Pr(X \leq x, Y \leq y).$$

Based on Sklar's theorem,[25] there is a copula C such that

$$H(x, y) = C(F(x), G(y)), \quad x, y \in R.$$

The theorem indicates that copula is a joint distribution function, and a joint distribution function can also be presented as copula given its marginal distributions. Schweizer[27] discussed that the joint distribution modeling can be reduced to copula modeling. Since copula represents the dependency of variables, it is also called dependence function.

There are several copulas that can be selected to study the dependency of variables. The choice of most appropriate copula has been an important issue. Due to its familiarity, Gaussian copula[28] was the most famous one compared to others; however, it failed to capture heavy-tail dependency, nonlinearity and asymmetry. Therefore, alternative copulas have been used to model joint dependences.

Gumbel copula[29] is able to capture right-tail dependence which is a particularly explored aspect of default dependency. Clayton copula[30] is defined to have left-tail dependence. Student's t copula[31] captures the symmetric-tail dependence with equal right- and left-tail dependences, while Gaussian and Frank copulas[32] are defined as symmetric dependences without any tail dependence.

2.2. Cyclostationary Time Series

In the classical time series analysis, the stationarity of time series is an important assumption. But in many real-world problems such as climatology and hydrology, datasets have cyclic rhythm and consequently the stationarity condition is not satisfied. As an efficient choice, CS processes[33,34] are employed to model the datasets with cyclic rhythms.

A process X_t is CS with period T (CS $-$ T) if T is the minimum natural number so that

$$m(t) := E(X_t) = m(t+T)$$

and

$$\begin{aligned} R(s,t) &:= \mathrm{Cov}(X_s, X_t) \\ &= E[(X_s - m(s))\overline{(X_t - m(t))}] \\ &= \mathrm{Cov}(X_{s+T}, X_{t+T}), \end{aligned}$$

for all $st \in Z$.

To detect the cyclic correlation, the coherent statistic[34] can be employed. The coherent statistic has been defined by

$$|\hat{\gamma}(p,q,M)|^2 = \frac{|\sum_{m=0}^{M-1} d_X(\lambda_{p+m})\overline{d_X(\lambda_{q+m})}|^2}{\sum_{m=0}^{M-1}|d_X(\lambda_{p+m})|^2 \sum_{m=0}^{M-1}|d_X(\lambda_{q+m})|^2}, \quad p > 0, \quad q \leq N$$

such that

$$d_X(\lambda) = n^{-1/2}\sum_{t=1}^{n} X_t e^{i(t-1)\lambda}, \quad \lambda \in [0, 2\pi),$$

is the discrete Fourier transform (DFT) of the observations $X_1, \ldots, X_n$, M is the parameter of smoothness and

$$(\lambda_i, \lambda_j) \in \bigcup_{k\in Z}\left\{(\lambda_i,\lambda_j) \in [0,2\pi)^2 : \lambda_j = \lambda_i + \frac{2\pi k}{T}\right\}.$$

Since the spectral domain of CS $-$ T time series is supported on the lines

$$T_j(x) = x \pm \frac{2\pi(j-1)}{T}, \quad j = 1, \ldots, T$$

it is expected that the plot of its coherent statistic emphasizes these lines.

Because of cyclic rhythms in the CS processes, using the usual copula regression to evaluate the relationship between two CS$-T$ time series is somewhat wrong. To solve this issue, in this research, a new approach is employed to evaluate the relationship between two CS $-$ T time series.

2.3. Fractional Brownian Motion

An fBm with Hurst index $H \in (0,1)$ is defined by

$$B_H(t) = \frac{1}{\Gamma\left(H + \frac{1}{2}\right)}\int_0^t (t-s)^{H-\frac{1}{2}} dB(s),$$

such that Γ and B are the gamma function and Brownian motion process,[35,36] respectively.

2.4. Copula for Cyclostationary Time Series

As previously discussed, the copula technique is most efficient for the stationary time series and may not work well for nonstationary time series such as the cyclostationary time series. To solve this issue, in this research, we introduce cyclocopula technique that is a copula-based regression method.

Assume X_t and Y_t are two CS $-$ T time series. Suppose $\{x_1, \ldots, x_n\}$ is a path of X_t and $\{y, \ldots, y_n\}$ is a path of Y_t. Without loss of generality, assume n is a multiplier of T $\quad n = mTm \in N$ The outline of cyclocopula technique is as follows:

(i) Split $\{x_1, \ldots, x_n\}\{y_1, \ldots, y_n\}$ and $\{(x_1, y_1), \ldots, (x_n, y_n)\}$ into T partitions $\{x^{(1)}, \ldots, x^{(T)}\}$, $\{y^{(1)}, \ldots, y^{(T)}\}$ and $\{(x,y)^{(1)}, \ldots, (x,y)^{(T)}\}$, such that

$$x^{(i)} = \{x_i, x_{i+T}, \ldots, x_{i+(m-1)T}\}, \quad i = 1, \ldots, T,$$

$$y^{(i)} = \{y_i, y_{i+T}, \ldots, y_{i+(m-1)T}\}, \quad i = 1, \ldots, T$$

and

$$\begin{aligned} (x,y)^{(i)} = \{&(x_i, y_i), (x_{i+T}, y_{i+T}), \ldots, \\ &(x_{i+(m-1)T}, y_{i+(m-1)T})\}, \\ &\qquad i = 1, \ldots, T. \end{aligned}$$

(ii) Let F_i, G_i and H_i be the distribution functions of the members of $x^{(i)}, y^{(i)}$ and $(x,y)^{(i)}$, respectively. Define the copula of $x^{(i)}$ and $y^{(i)}$

by

$$C_i = C(F_i, G_i), \quad i = 1, \ldots T,$$

and estimate the copula of $x^{(i)}$ and $y^{(i)}$ by

$$\hat{C}_i = \hat{C}(F_i, G_i), \quad i = 1, \ldots, T.$$

(iii) Apply copula regression to find the regression equation of $y^{(i)}$ based on $x^{(i)}$

$$\hat{y}_j = b_{0,i} + b_{0,i} x_j, \quad i = 1, \ldots, T,$$
$$j = i, i+T, i+(m-1)T.$$

(iv) Combine the regression equations for the next uses such as the prediction or goodness-of-fit tests.

Remark 1. If X_t and Y_t are two CS time series with different periods T_1 and T_2, let $T = \mathrm{lcm}(T_1, T_2)$ where lcm is the least common multiple of T_1 and T_2.

3. SIMULATION

In this section, the ability of the proposed method to detect the relationship between two CS time series is studied. For this purpose, numerous datasets from two CS time series X_t and Y_t are produced and analyzed.

The free software package of R software[37] version 4.0.5 running on a Core^{TM} i7-760 processor @ 2.8 GHz is used to implement the algorithms.

Assume the processes

$$X_t = \frac{1 + \alpha\cos(2\pi t/T)}{2} X_{t-1} + B_H(t)$$

and

$$Y_t = \beta X_t + W_t, \quad \{W_t\} \sim \mathrm{IIDN}(0, 1).$$

The simulation procedure is as follows:

Step 1. For the fixed $n \in \{120, 240, 480, 1200\}$, $\alpha, \beta, H \in \{0.1, 0.2, \ldots, 0.9\}$ and $T \in \{2, 3, 4\}$, separate paths of size n from two $\mathrm{CS} - T$ time series X_t and Y_t are generated.

Step 2. The simulated dataset is split into T partitions $\{x^{(1)}, \ldots, x^{(T)}\}$ and $\{y^{(1)}, \ldots, y^{(T)}\}$. Then the copula of $x^{(i)}$ and $y^{(i)}$ is estimated by

$$\hat{C}_i = \hat{C}(F_i, G_i), \quad i = 1, \ldots, T$$

In this study, five different copula families have been applied, as discussed in the following:

(1) *Gaussian copula*
The Gaussian copula is given by

$$C(a, b) = \frac{1}{\sqrt{1-\theta^2}} \times \exp\left(-\frac{(u^2+v^2)\theta^2 - 2uv\theta}{2(1-\theta^2)}\right), \quad \theta \in [-1, 1],$$

where

$$u = \sqrt{2}\mathrm{erf}^{-1}(2a - 1),$$
$$v = \sqrt{2}\mathrm{erf}^{-1}(2b - 1)$$

and

$$\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z \exp(-t^2) dt.$$

(2) *The t copula*
The t copula is represented as follows:

$$C(a, b) = T_{\theta,v}(t_v^{-1}(a), t_v^{-1}(b)),$$

where $T_{\theta,v}$ is the standardized bivariate Student's t distribution with correlation $\theta \in [-1, 1]$ and v degrees of freedom. $t_v^{-1}(\cdot)$ indicates the inverse of Student's t cumulative distribution function.

(3) *Clayton copula*
The Clayton copula is presented as

$$C(a, b) = (\max(a^{-\theta} + b^{-\theta} - 1, 0))^{-1/\theta}, \quad \theta \geq -1, \quad \theta \neq 0.$$

(4) *Gumbel copula*
The Gumbel copula has the following form:

$$C(a, b) = \exp(-((-\log a)^\theta + (-\log b)^\theta)^{\frac{1}{\theta}}) \quad \theta \geq 1.$$

(5) *Frank copula*
The Frank copula is denoted by

$$C(a, b) = -\frac{1}{\theta} \log\left(1 + \frac{(\exp(-\theta a) - 1) \times (\exp(-\theta b) - 1)}{\exp(-\theta) - 1}\right), \quad \theta \neq 0.$$

Step 3. For each copula, the regression analysis is applied to estimate the equation of $y^{(i)}$ based on $x^{(i)}$

$$\hat{y}_j = b_{0,i} + b_{0,i}x_j, \quad i = 1, \ldots, T,$$
$$j = i, i + T, i + (m-1)T.$$

Step 4. For each copula, the estimated copula regression equations are used to estimate the T partitions $\{y^{(1)}, \ldots, y^{(T)}\}$ by $\{\hat{y}^{(1)}, \ldots, \hat{y}^{(T)}\}$.

Step 5. Different goodness-of-fit measures including the correlation coefficient (r), Willmott's Index (WI) and Nash–Sutcliffe (NS) coefficient are computed as follows:

$$r = \frac{\sum_{i=1}^{n}(y_i - \bar{y})(\hat{y}_i - \bar{\hat{y}})}{\sqrt{\sum_{i=1}^{n}(y_i - \bar{y})^2}\sqrt{\sum_{i=1}^{n}(\hat{y}_i - \bar{\hat{y}})^2}},$$

$$\mathrm{WI} = \frac{\sum_{i=1}^{n}(y_i - \hat{y}_i)^2}{\sum_{i=1}^{n}(|y_i - \bar{y}| + |\hat{y}_i - \bar{y}|)^2},$$

and

$$\text{NS coefficient} = 1 - \frac{\sum_{i=1}^{n}(y_i - \hat{y}_i)^2}{\sum_{i=1}^{n}(\hat{y}_i - \bar{\hat{y}})^2}.$$

Step 6. Steps 1–5 are repeated 1000 times.

Step 7. For each parameter setting and each copula, the values of r WI and NS coefficient of all 1000 runs are computed.

To compare the proposed work with previous studies, for each parameter setting, the suitable simple linear regression and usual copula models are also fitted on the simulated dataset.

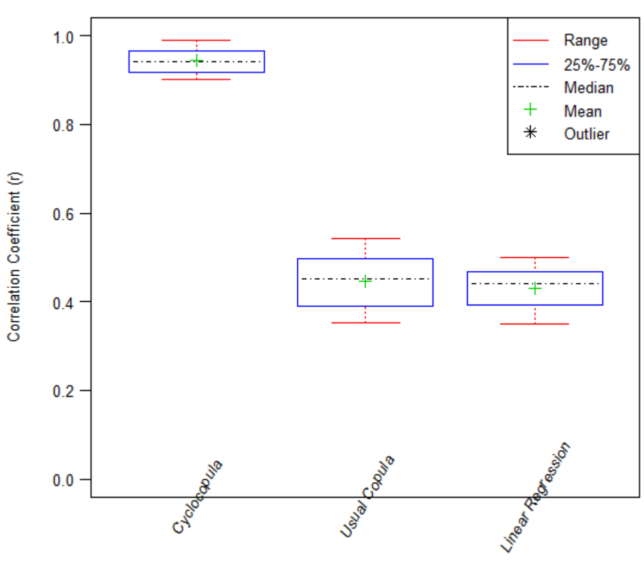

Fig. 1 The performances of the cyclocopula, usual copula and linear regression techniques in terms of the correlation coefficient (r).

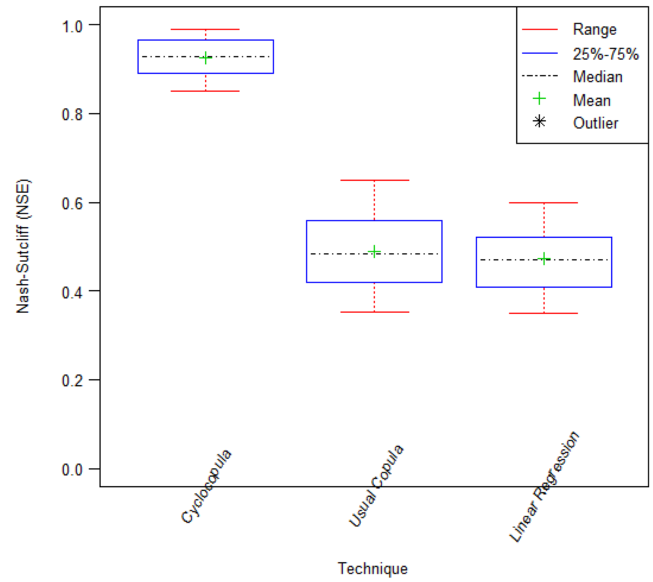

Fig. 2 The performances of the cyclocopula, usual copula and linear regression techniques in terms of the NS coefficient.

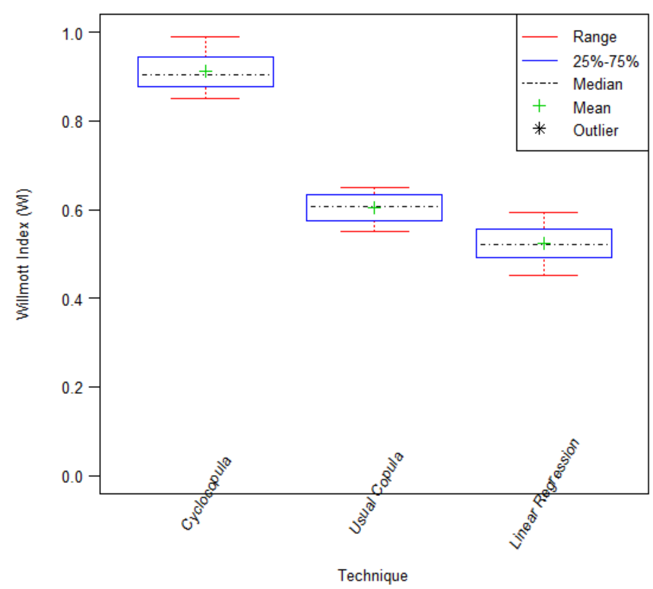

Fig. 3 The performances of the cyclocopula, usual copula and linear regression techniques in terms of WI.

Figures 1–3 demonstrate the error-bar charts of goodness-of-fit indices for different fitted models. As it can be seen, for the cyclocopula technique, the values of goodness-of-fit indices are too close to one. Moreover, the fitted cyclocopula model has the maximum values of r WI and NS coefficient among all fitted models. In other words, the proposed method is robust to detect the relationship between two cyclostationary time series with fBm errors.

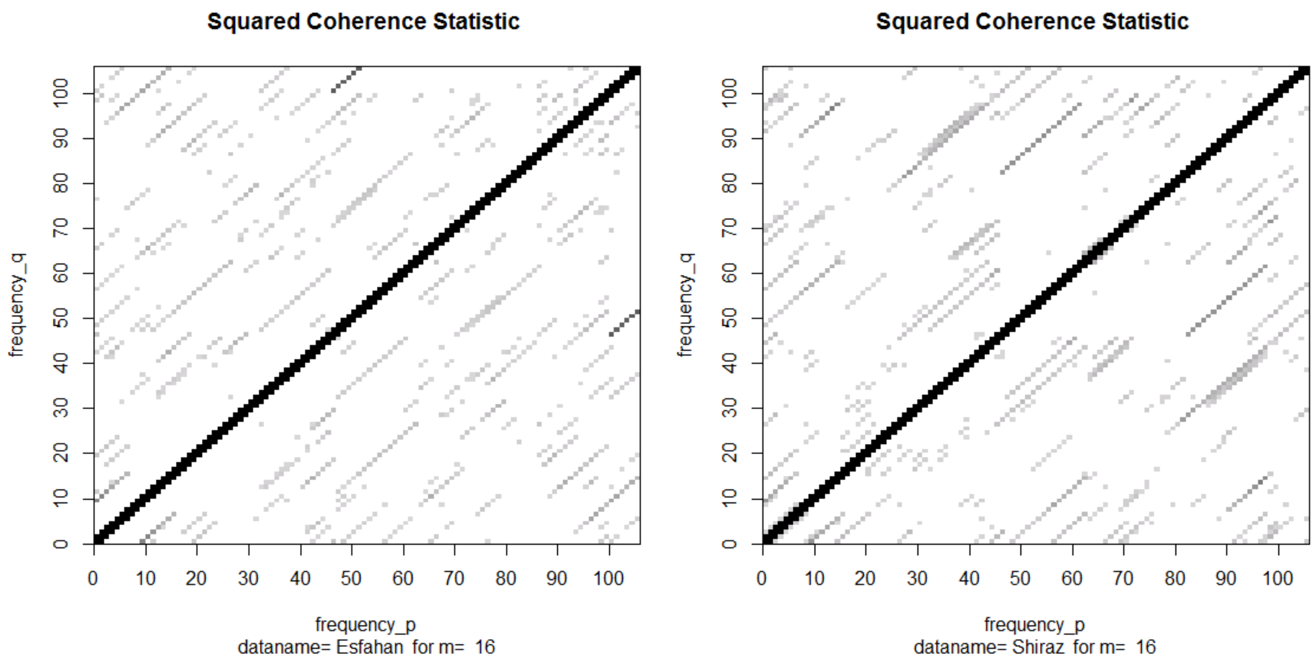

Fig. 4 Spectral supports for the seasonal RDI datasets (left: Esfahan; right: Shiraz).

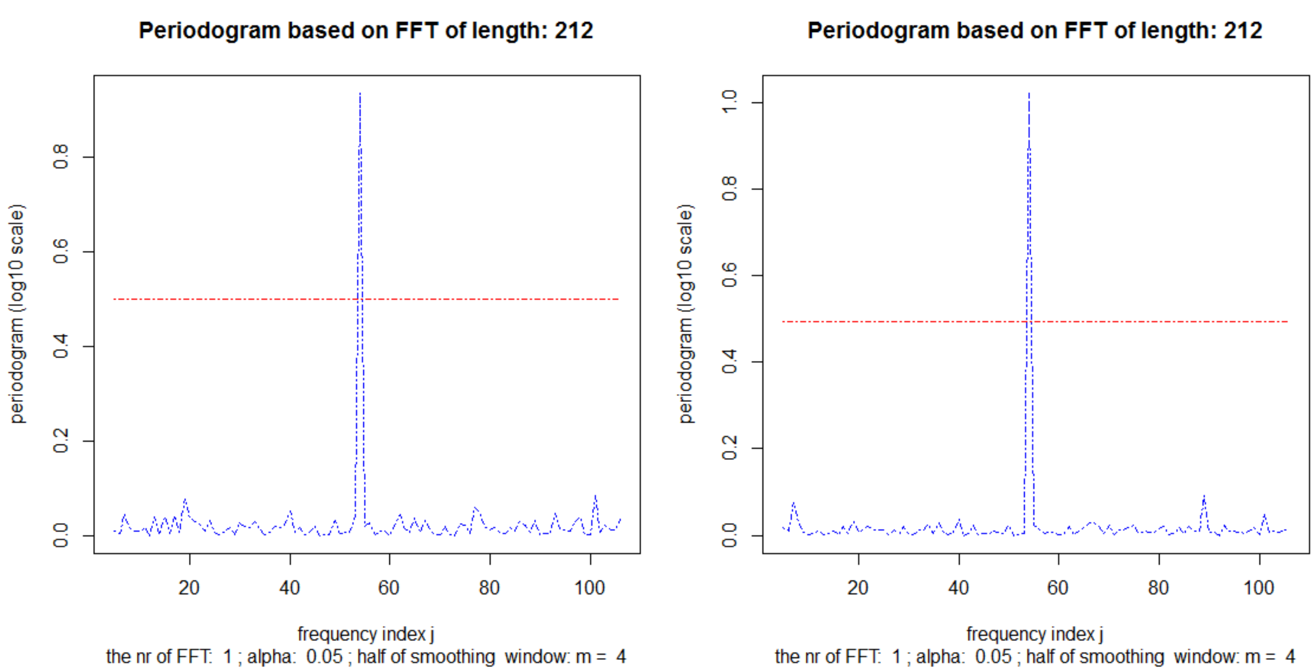

Fig. 5 Periodograms for the seasonal RDI datasets (left: Esfahan; right: Shiraz).

4. REAL DATA

In this section, the applicability of cyclocopula approach in practice is studied by a real example. For this purpose, the seasonal Reconnaissance Drought Indexes (RDIs, in short)[38–40] of two Iranian synoptic stations (Esfahan and Shiraz) from 1967 to 2019 are considered. As it can be seen in Figs. 4 and 5, the spectral domains of these datasets are supported by the lines $T_j(x) = x \pm \frac{2\pi j}{4} j = 0, \ldots, 3$, and the peak of periodogram is at 53. Therefore, it can be confirmed that the datasets detect cyclostationary processes with period 4 ($\frac{212}{53} = 4, p < 0.05$).

To evaluate the performance of the presented method, the cyclocopula technique is applied to

Table 1 The Values of MSE and R^2 for Different Fitted Models.

Method	MSE	R^2
Cyclocopula	0.019	0.989
Copula	0.162	0.661
Simple linear regression	0.191	0.592

detect the relationship between the RDIs of Esfahan and Shiraz synoptic stations. To compare the proposed work with previous studies, suitable simple linear regression and the usual copula models are also fitted. Table 1 summarizes the values of R^2 and the mean-square error (MSE) of the different fitted models. As Table 1 indicates, the fitted cyclocopula model has the minimum MSE and the maximum R^2 among all fitted models. Consequently, the presented approach has the best performance in detecting the relationship between the RDIs of Esfahan and Shiraz synoptic stations.

5. CONCLUSION

For determining the relationship between two stationary time series, cross-correlation function, simple linear regression and copula models are suggested. Cross-correlation function and simple linear regression are sensitive to the existence of outliers and abnormality of datasets. An efficient alternative way to model the dependency between two stationary time series is to use copula. Copula technique is most efficient for the stationary time series and may not work well for nonstationary time series such as the cyclostationary time series. To solve this issue, in this research, we introduced a copula-based regression technique, called cyclocopula. The ability of the proposed approach to detect the relationship between two cyclostationary time series with fractional Brownian motion errors was studied. For this purpose, numerous datasets were produced and analyzed. The results indicated that the values of goodness-of-fit indices were close to one for cyclocopula approach. Moreover, the fitted cyclocopula model had the best performance compared with other alternative approaches. The applicability of cyclocopula approach in practice was also studied by a real example involving the seasonal RDIs of two Iranian synoptic stations (Esfahan and Shiraz). Several methods were applied to detect the relationship between the RDIs of Esfahan and Shiraz synoptic stations. The fitted cyclocopula model had the minimum MSE and the maximum R^2 among all fitted models. Consequently, the presented approach had the best performance in detecting the relationship between the RDIs of Esfahan and Shiraz synoptic stations.

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