Annals of Pure and Applied Logic 173 (2022) 103100

Contents lists available at ScienceDirect

Annals of Pure and Applied Logic

www.elsevier.com/locate/apal

Complexity in the interdefinability of timelike, lightlike and spacelike relatedness of Minkowski spacetime [☆]

Hajnal Andréka [a], Judit X. Madarász [a], István Németi [a], Gergely Székely [a,b,*]

[a] Alfréd Rényi Institute of Mathematics, H-1053 Budapest, Reáltanoda utca 13-15, Budapest, Hungary
[b] University of Public Service, H-1083 Budapest, 2 Ludovika tér, Budapest, Hungary

ARTICLE INFO

Article history:
Received 10 January 2022
Received in revised form 27 January 2022
Accepted 27 January 2022
Available online 2 February 2022

MSC:
03B30
83A05
03B10
03C40
03G99

Keywords:
First-order logic
Relativity theory
Definable concepts
Minkowski spacetime
Quantifier complexity
Definability theory

ABSTRACT

Interdefinability of timelike, lightlike and spacelike relatedness of Minkowski spacetime is investigated in detail in the paper, with the aim of finding the simplest definitions. Based on ideas scattered in the literature, definitions are given between any two of these binary relations that use 4 variables, i.e., they use only 2 auxiliary variables. All these definitions work over arbitrary Euclidean fields in place of the field of reals, if the dimension n of spacetime is greater than two. If $n = 2$, the definitions work over arbitrary ordered fields except the ones based on lightlike relatedness (where no definition can work by symmetry). None of these relations can be defined from another one using only one auxiliary variable. These definitions use only one universal and one existential quantifiers in a specific order. In some of the cases, we show that the order of these quantifiers can be reversed for the price of using twice as many quantifiers. Except in two cases, we provide existential/universal definitions using 3 auxiliary variables or show that no existential/universal definition exists. There are no existential/universal definitions between any two of these relations using only 2 auxiliary variables. It remains open whether there is an existential (universal) definition of timelike (lightlike) relatedness from spacelike relatedness if $n > 2$. Finally, several other open problems related to the quantifier complexity of the simplest possible definitions are given.

1. Introduction

There is an extensive literature on the axiomatization of Minkowski spacetime in terms of various basic concepts. Robb [14], revised as [15], gives an axiomatization of Minkowski spacetime using only one binary "after" relation as primitive notion. Goldblatt [4, Appendix B] shows that "between" and "orthogonali-

☆ This research is supported by the Hungarian National Research, Development and Innovation Office (NKFIH), grants no. FK-134732 and TKP2021-NVA-16.

* Corresponding author at: Alfréd Rényi Institute of Mathematics, H-1053 Budapest, Reáltanoda utca 13-15, Budapest, Hungary.
E-mail addresses: andreka.hajnal@renyi.hu (H. Andréka), madarasz.judit@renyi.hu (J.X. Madarász), nemeti.istvan@renyi.hu (I. Németi), szekely.gergely@renyi.hu (G. Székely).

https://doi.org/10.1016/j.apal.2022.103100

ty", the primitives used therein, can be defined in terms of Robb's "after" relation. Latzer [8] shows that Minkowski spacetime can be axiomatized using the relation of lightlike relatedness alone. Both in terms of the binary asymmetric lightlike after relation and in terms of the binary symmetric lightlike relatedness relation, Mundy [12] axiomatizes Minkowski spacetime significantly simplifying Robb's axiom system. Pambuccian [13] explicitly defines collinearity and equidistance from lightlike relatedness in metric-affine Fano spaces in order to present Alexandrov–Zeeman type theorems as definability results. For a contemporary axiomatization of Minkowski spacetime pursued in the style of Tarski, see [3].

The three most fundamental symmetric binary relations in Minkowski spacetime are timelike, lightlike and spacelike relatedness. Scattered in the extensive literature mentioned above, there can be found some definitions and some claims about certain interdefinability among these relations. For example, Malament [10, p. 294] claims that the three basic relations are first-order definable in terms of one another, but he gives no proof or reference except for one specific formula in footnote 3. Interdefinability results concerning Minkowski spacetime are used in general relativity, see e.g., [7], [5, especially p. 180] and [11]. The results in the present paper are used in the follow-up paper [1].

Herein we carefully investigate the interdefinability of timelike, lightlike and spacelike relatedness relations aiming to find the simplest definitions. For each pair of these relations, we present definitions that are simplest in terms of number of variables. We also investigate how definability depends on the dimension of spacetime and on the underlying ordered field. For example, lightlike relatedness is definable from timelike relatedness even if the dimension n of spacetime is 2 (i.e., not just time but space is also one dimensional) but n has to be at least 3 to be able to define timelike relatedness from lightlike relatedness. In most of the cases, we assume that either $n = 2$ and the underlying field is an arbitrary ordered field or $n > 2$ and the underlying field is a Euclidean field (i.e., an ordered field in which every positive number has a square root).

In Section 3, we show that the relations of timelike, lightlike and spacelike relatedness can be defined from any of them using four variables, i.e., using only two auxiliary variables, see Theorems 1, 2 and 3. Then we show that the number of variables used in these definitions are minimal, i.e., none of these relations can be defined from another one using only one auxiliary variable, see Theorem 4. In Section 4, using a well-visualizable construction, we show that spacelike relatedness can be defined existentially from timelike relatedness using four auxiliary variables, see Theorem 5, but no such definitions exist using only two auxiliary variables cf. Theorem 7. In preparation to the proof of Theorem 7, we develop a picturesque graph-embedding-based method to understand relations definable by existential formulas. Then using this method, we give an existential formula defining spacelike relatedness from timelike relatedness that uses only three auxiliary variables, see Theorem 8. Over arbitrary ordered fields, we show that lightlike relatedness cannot be defined existentially neither from timelike nor from spacelike relatedness, see Theorem 9. Over arbitrary ordered fields, we show that neither timelike nor spacelike relatedness can be defined existentially or universally from lightlike relatedness, see Theorem 10. Finally in Section 5, we give several open problems related to the quantifier complexity of the simplest possible definitions.

2. Some notations and definitions

Here, we collect the most fundamental notations used in this paper. We work over arbitrary ordered fields. So herein, we assume that $(Q, +, \cdot, \leq)$ is an ordered field.[1] Let Q^n denote the set of n-dimensional vectors over Q. We also use the vector space structure on Q^n. Following the convention common in relativity theory, we start counting axes from 0, draw the 0th coordinate axis vertically and consider it as the time-axis.

[1] That $(Q, +, \cdot, \leq)$ is an ordered field means that $(Q, +, \cdot)$ is a field which is totally ordered by $\leq$, and we have the following two properties for all $x, y, z \in Q$: (1) $x + z \leq y + z$ if $x \leq y$, and (2) $0 \leq xy$ if $0 \leq x$ and $0 \leq y$.

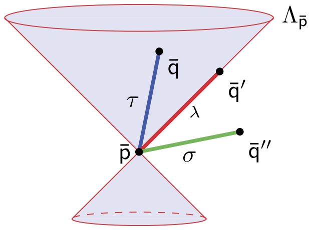

Fig. 1. Illustration for relations $\bar{\mathsf{p}}\ \tau\ \bar{\mathsf{q}}$, $\bar{\mathsf{p}}\ \lambda\ \bar{\mathsf{q}}'$, and $\bar{\mathsf{p}}\ \sigma\ \bar{\mathsf{q}}''$, as well as for light cone $\Lambda_{\bar{\mathsf{p}}}$.

In our formulas, we use the following symbols for logical connectives: both "$\wedge$" and "," interchangeably for conjunction, "$\vee$" for disjunction, "$\neg$" for negation, "$\exists$" for existential quantifier, and "$\forall$" for universal quantifier.

Let $\bar{\mathsf{p}} = (\mathsf{p}_0, \mathsf{p}_1, \dots, \mathsf{p}_{n-1}) \in Q^n$ and $\bar{\mathsf{q}} = (\mathsf{q}_0, \mathsf{q}_1, \dots, \mathsf{q}_{n-1}) \in Q^n$. We are going to work with the following relations illustrated by Fig. 1:

$$\bar{\mathsf{p}}\ \tau\ \bar{\mathsf{q}} \overset{\text{def}}{\iff} (\mathsf{p}_0 - \mathsf{q}_0)^2 > (\mathsf{p}_1 - \mathsf{q}_1)^2 + \ldots + (\mathsf{p}_{n-1} - \mathsf{q}_{n-1})^2$$
$$\bar{\mathsf{p}}\ \lambda\ \bar{\mathsf{q}} \overset{\text{def}}{\iff} (\mathsf{p}_0 - \mathsf{q}_0)^2 = (\mathsf{p}_1 - \mathsf{q}_1)^2 + \ldots + (\mathsf{p}_{n-1} - \mathsf{q}_{n-1})^2\ ,\ \mathsf{p}_0 \neq \mathsf{q}_0$$
$$\bar{\mathsf{p}}\ \sigma\ \bar{\mathsf{q}} \overset{\text{def}}{\iff} (\mathsf{p}_0 - \mathsf{q}_0)^2 < (\mathsf{p}_1 - \mathsf{q}_1)^2 + \ldots + (\mathsf{p}_{n-1} - \mathsf{q}_{n-1})^2.$$

We say that $\bar{\mathsf{p}}$ *and* $\bar{\mathsf{q}}$ *are*, ***timelike***, ***lightlike***,[2] ***spacelike related*** in the respective cases. A line is called timelike (lightlike, spacelike) iff all of its distinct points are timelike (lightlike, spacelike) related. By that $\bar{\mathsf{p}}$ is in the ***timelike future*** *of* $\bar{\mathsf{q}}$, we simply mean that $\bar{\mathsf{p}}\ \tau\ \bar{\mathsf{q}}$ and $\mathsf{p}_0 > \mathsf{q}_0$. Analogously, $\bar{\mathsf{p}}$ is in the ***timelike past*** *of* $\bar{\mathsf{q}}$ iff $\bar{\mathsf{p}}\ \tau\ \bar{\mathsf{q}}$ and $\mathsf{p}_0 < \mathsf{q}_0$.

We denote the complement of these relations by $\bar{\tau}$, $\bar{\lambda}$, $\bar{\sigma}$. By $\rho_{\neq}$, we are going to abbreviate the intersection of binary relations ρ and $\neq$. In our formulas, we use the same symbols τ, λ and σ in infix notation also to denote the corresponding relation symbols. We abbreviate negated atomic formulas $\neg(x = y)$ to $x \neq y$, $\neg(x\ \tau\ y)$ to $x\ \bar{\tau}\ y$, etc. to make them easier to read. In case $\varphi(x, y)$ is a formula having all its free variables among x and y, we will write $\varphi(\bar{\mathsf{x}}, \bar{\mathsf{y}})$ to denote that the binary relation defined by φ holds for $\bar{\mathsf{x}}, \bar{\mathsf{y}} \in Q^n$.

By *binary relation R on the universe of model $\mathfrak{M}$ is* ***definable*** *in $\mathfrak{M}$* ***by using k many auxiliary variables*** we mean the following: there is a formula $\varphi(x, y)$ in the language of $\mathfrak{M}$ and variables z_1, z_2, …, z_k such that $x, y, z_1, \dots, z_k$ are all the variables that occur in φ and φ defines relation R in $\mathfrak{M}$, in the sense that the expansion of $\mathfrak{M}$ with R validates the formula $\forall x \forall y[R(x, y) \leftrightarrow \varphi(x, y)]$.

By the ***light cone*** $\Lambda_{\bar{\mathsf{p}}}$ *through point* $\bar{\mathsf{p}}$, we understand the set of points which are lightlike related or equal to $\bar{\mathsf{p}}$, i.e.,

$$\Lambda_{\bar{\mathsf{p}}} \overset{\text{def}}{=} \left\{\bar{\mathsf{q}} \in Q^n : \bar{\mathsf{q}}\ \lambda\ \bar{\mathsf{p}} \text{ or } \bar{\mathsf{p}} = \bar{\mathsf{q}}\right\}.$$

By the ***causal cone*** $\mathsf{C}_{\bar{\mathsf{p}}}$ *through point* $\bar{\mathsf{p}}$, we understand the set of points which are equal, timelike or lightlike (aka. causally) related to $\bar{\mathsf{p}}$, i.e.,

$$\mathsf{C}_{\bar{\mathsf{p}}} \overset{\text{def}}{=} \left\{\bar{\mathsf{q}} \in Q^n : \bar{\mathsf{q}}\ \bar{\sigma}\ \bar{\mathsf{p}}\right\},$$

and $\bar{\mathsf{p}}$ is in the ***causal future (past)*** *of* $\bar{\mathsf{q}}$ iff $\bar{\mathsf{p}}\ \bar{\sigma}\ \bar{\mathsf{q}}$ and $\mathsf{p}_0 \geq \mathsf{q}_0$ ($\mathsf{p}_0 \leq \mathsf{q}_0$).

[2] Usually, it is allowed for lightlike related points to be equal, see e.g., [10]. Here we forbid equality by assuming $\mathsf{p}_0 \neq \mathsf{q}_0$. From the point of view of definability, this is a negligible difference since the two versions are clearly definable from each other. We prefer to use the strict one because that is minimal among the definable nonempty binary concepts.

Remark 1 *($n \geq 2$).* The following is known and also straightforward to check. For all natural numbers $n \geq 2$, the followings are all automorphisms of model $(Q^n, \tau, \lambda, \sigma)$: any uniform scaling, reversing time,[3] any translation, any spatial rotation,[4] any Lorentz boost[5] in any spatial direction.

An ordered field $(Q, +, \cdot, \leq)$ is called a ***Euclidean field*** if its every positive element has a square root, i.e., iff the following holds in it

$$\text{for all } x \in Q, \text{ if } 0 \leq x, \text{ there is a } y \in Q \text{ such that } x = y^2. \tag{Eucl.}$$

In figures and tables, we are going to refer to this property as (Eucl.). In some proofs, we will refer to plane $\{(\mathsf{t}, \mathsf{x}, 0, \ldots, 0) : \mathsf{t}, \mathsf{x} \in Q\}$ as tx-***plane***.

Proposition 1 *($n \geq 2$; (Eucl.) or $n = 2$). If $n = 2$ or $(Q, +, \cdot, \leq)$ is a Euclidean field, then the automorphism group of $(Q^n, \tau, \lambda, \sigma)$ acts transitively on the timelike, lightlike and spacelike related pairs of points, i.e., for all $\rho \in \{\tau, \lambda, \sigma\}$ and for all $\bar{\mathsf{p}}, \bar{\mathsf{q}}, \bar{\mathsf{p}}', \bar{\mathsf{q}}' \in Q^n$ for which both $\bar{\mathsf{p}}\ \rho\ \bar{\mathsf{q}}$ and $\bar{\mathsf{p}}'\ \rho\ \bar{\mathsf{q}}'$ holds, there is an automorphism α of $(Q^n, \tau, \lambda, \sigma)$ for which $\alpha(\bar{\mathsf{p}}) = \bar{\mathsf{p}}'$ and $\alpha(\bar{\mathsf{q}}) = \bar{\mathsf{q}}'$.*

Proof. Because the inverses and the compositions of automorphisms are automorphisms, it is enough to show that for every two distinct points $\bar{\mathsf{p}}, \bar{\mathsf{q}} \in Q^n$ there is an automorphism α of $(Q^n, \tau, \lambda, \sigma)$ such that α maps $\bar{\mathsf{p}}$ to $(0, 0, \ldots, 0)$ and $\alpha(\bar{\mathsf{q}}) = (1, 0, \ldots, 0)$ or $\alpha(\bar{\mathsf{q}}) = (1, 1, 0, \ldots, 0)$ or $\alpha(\bar{\mathsf{q}}) = (0, 1, 0, \ldots, 0)$.

Since translations are automorphisms of $(Q^n, \tau, \lambda, \sigma)$, cf. Remark 1, there is an automorphism which maps $\bar{\mathsf{p}}$ to $(0, 0, \ldots, 0)$. So because the composition of two automorphisms is also an automorphism, we can assume that $\bar{\mathsf{p}} = (0, 0, \ldots, 0)$ without loosing generality. Similarly, because reversing time is also an automorphism, cf. Remark 1, we can also assume that $\mathsf{q}_0 \geq 0$.

Assume now that $(Q, +, \cdot, \leq)$ is a Euclidean field. Then there is a spatial rotation R which does not change $\bar{\mathsf{p}} = (0, 0, \ldots, 0)$ and maps $\bar{\mathsf{q}}$ to $(\mathsf{t}, \mathsf{x}, 0, \ldots, 0)$ for some non-negative $\mathsf{t}, \mathsf{x} \in Q$. This rotation R is an automorphism, cf. Remark 1. Hence we can assume that $\bar{\mathsf{p}} = (0, 0, \ldots, 0)$ and $\bar{\mathsf{q}} = (\mathsf{t}, \mathsf{x}, 0, \ldots, 0)$ for some $\mathsf{t} \geq 0$, $\mathsf{x} \geq 0$ for which $(\mathsf{t}, \mathsf{x}) \neq (0, 0)$.

If $\mathsf{t} > \mathsf{x} \geq 0$, then $\bar{\mathsf{p}}\ \tau\ \bar{\mathsf{q}}$. Let us consider the following map:

$$B_\tau : (\mathsf{r}_0, \mathsf{r}_1, \mathsf{r}_2, \ldots, \mathsf{r}_{n-1}) \mapsto \frac{1}{\sqrt{\mathsf{t}^2 - \mathsf{x}^2}} \left(\frac{\mathsf{t}\mathsf{r}_0 - \mathsf{x}\mathsf{r}_1}{\sqrt{\mathsf{t}^2 - \mathsf{x}^2}}, \frac{\mathsf{t}\mathsf{r}_1 - \mathsf{x}\mathsf{r}_0}{\sqrt{\mathsf{t}^2 - \mathsf{x}^2}}, \mathsf{r}_2, \ldots, \mathsf{r}_{n-1} \right).$$

This map B_τ is an automorphism because it is the composition of a Lorentz boost corresponding to speed $\mathsf{v} = \mathsf{x}/\mathsf{t}$ and a uniform scaling by the factor $1/\sqrt{\mathsf{t}^2 - \mathsf{x}^2}$. It is straightforward to verify that B_τ maps $\bar{\mathsf{p}}$ to $(0, 0, \ldots, 0)$ and $\bar{\mathsf{q}}$ to $(1, 0, \ldots, 0)$. If $\mathsf{t} = \mathsf{x} > 0$, then $\bar{\mathsf{p}}\ \lambda\ \bar{\mathsf{q}}$ and there is a uniform scaling that maps $\bar{\mathsf{p}}$ to $(0, 0, \ldots, 0)$ and $\bar{\mathsf{q}}$ to $(1, 1, 0, \ldots, 0)$. If $0 < \mathsf{t} < \mathsf{x}$, then $\bar{\mathsf{p}}\ \sigma\ \bar{\mathsf{q}}$ and let

$$B_\sigma : (\mathsf{r}_0, \mathsf{r}_1, \mathsf{r}_2, \ldots, \mathsf{r}_{n-1}) \mapsto \frac{1}{\sqrt{\mathsf{x}^2 - \mathsf{t}^2}} \left(\frac{\mathsf{x}\mathsf{r}_0 - \mathsf{t}\mathsf{r}_1}{\sqrt{\mathsf{x}^2 - \mathsf{t}^2}}, \frac{\mathsf{x}\mathsf{r}_1 - \mathsf{t}\mathsf{r}_0}{\sqrt{\mathsf{x}^2 - \mathsf{t}^2}}, \mathsf{r}_2, \ldots, \mathsf{r}_{n-1} \right).$$

[3] That is, the transformation $(\mathsf{p}_0, \mathsf{p}_1 \ldots, \mathsf{p}_{n-1}) \mapsto (-\mathsf{p}_0, \mathsf{p}_1 \ldots, \mathsf{p}_{n-1})$.

[4] That is, any distance-preserving linear transformation of determinant 1 that fixes the time-axis, i.e., maps vector $(1, 0, \ldots 0)$ to itself.

[5] A Lorentz boost in the i-th spatial coordinate direction is a transformation of the form

$$B_i : (\mathsf{p}_0, \mathsf{p}_1, \ldots, \mathsf{p}_{n-1}) \mapsto \left(\frac{\mathsf{p}_0 - \mathsf{v}\mathsf{p}_i}{\sqrt{1 - \mathsf{v}^2}}, \mathsf{p}_1, \ldots, \mathsf{p}_{i-1}, \frac{\mathsf{p}_i - \mathsf{v}\mathsf{p}_0}{\sqrt{1 - \mathsf{v}^2}}, \mathsf{p}_{i+1}, \ldots, \mathsf{p}_{n-1} \right)$$

for some velocity $\mathsf{v} \in Q$ for which $-1 < \mathsf{v} < 1$ and $\sqrt{1 - \mathsf{v}^2} \in Q$.

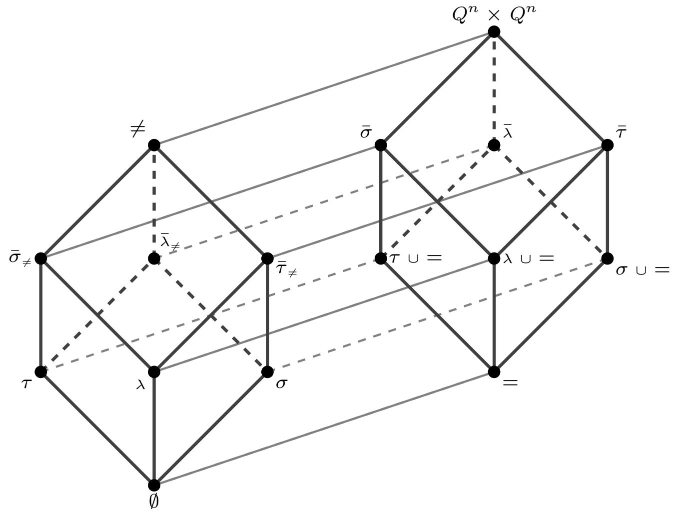

Fig. 2. The figure illustrates the Boolean algebra generated by relations τ, λ, σ, and $=$.

Analogously, this B_σ is an automorphism and maps $\bar{\mathsf{p}}$ to $(0, 0, \ldots, 0)$ and $\bar{\mathsf{q}}$ to $(0, 1, 0, \ldots, 0)$. If $\mathsf{t} = 0 < \mathsf{x}$, then $\bar{\mathsf{p}}\ \sigma\ \bar{\mathsf{q}}$ and there is a uniform scaling which maps $\bar{\mathsf{p}}$ and $\bar{\mathsf{q}}$ to $(0, 0, \ldots, 0)$ and $(0, 1, 0, \ldots, 0)$, respectively. This completes the proof in the case when $(Q, +, \cdot, \leq)$ is a Euclidean field.

There were only two steps where we used the Euclidean property: the one where we used the existence of an appropriate rotation R to rotate points $\bar{\mathsf{p}}$ and $\bar{\mathsf{q}}$ to the tx-plane, and the one where we used the existence of maps B_τ and B_σ. Since when $n = 2$, the rotation step is not needed and B_τ and B_σ simplify into rational functions in both coordinates, we also have the statement over arbitrary ordered fields if $n = 2$. $\square$

Because any relation definable in a model has to be closed under the automorphisms of the model, the following is a corollary of Proposition 1.

Corollary 1 *($n \geq 2$; (Eucl.) or $n = 2$). Let ρ and δ be any two different relations from the set $\{\tau, \lambda, \sigma\}$. For any relation R definable in (Q^n, ρ), if $n = 2$ or $(Q, +, \cdot, \leq)$ is a Euclidean field, we have that*

$$\delta \subseteq R \iff (p, q) \in R \text{ for some } p, q \in Q^n \text{ such that } p\ \delta\ q \iff R \cap \delta \neq \emptyset. \qquad \square$$

Because the union of relations τ, σ, λ and $=$ is the universal relation $Q^n \times Q^n$, Corollary 1 implies the following.

Corollary 2 *($n \geq 2$; (Eucl.) or $n = 2$). Let ρ be any of relations τ, λ and σ. If $n = 2$ or $(Q, +, \cdot, \leq)$ is a Euclidean field, then any nonempty binary relation which is definable in (Q^n, ρ) is the union of some of relations τ, σ, λ and $=$.* $\square$

In other words, Corollary 2 says that, assuming $n = 2$ or $(Q, +, \cdot, \leq)$ is a Euclidean field, if relations τ, σ and λ are definable in any of structures (Q^n, τ), (Q^n, λ) and (Q^n, σ), then they and $=$ are atoms in the corresponding Boolean algebra of definable binary concepts. We will see that they are all definable except in (Q^2, λ), see Remark 2 and Theorems 1, 2 and 3 in Section 3. So except in case (Q^2, λ), we have that the Boolean algebra of definable binary concepts is the 16 element one illustrated by Fig. 2. In case of (Q^2, λ), the algebra of definable binary concepts is the 8 element one generated by relations λ and $=$.

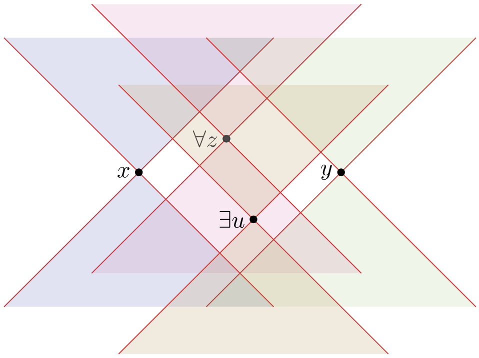

Fig. 3. Formula $\Psi_{\tau\to\sigma}(x, y)$, defining spacelike relatedness σ from timelike relatedness τ, intuitively says that x and y are distinct points and inside the light cone of every point z (different from x and y) there is a point u which is inside neither the light cone of x nor that of y.

3. Interdefinability using minimal number of variables

In this section, we are going to show that timelike, spacelike and lightlike relatedness are definable from one another using 2 auxiliary variables, but not definable using only one auxiliary variable. To do so, let us introduce first formula $\Psi_{\tau\to\sigma}$ defining spacelike relatedness σ from timelike relatedness τ:

$$\Psi_{\tau\to\sigma}(x, y) \stackrel{\text{def}}{=} x \neq y\,,\, \forall z\big(z = x \vee z = y \vee \exists u(u\;\tau\; z\,,\, u\;\bar\tau\; x\,,\, u\;\bar\tau\; y)\big).$$

Formula $\Psi_{\tau\to\sigma}(x, y)$ intuitively says that $x \neq y$ and inside the light cone of every point z (different from x and y) there is a point u which is inside neither the light cone of x nor that of y, see Fig. 3. Using this definition, we can easily define λ from τ by the following formula:

$$\Psi_{\tau\to\lambda}(x, y) \stackrel{\text{def}}{=} \neg\Psi_{\tau\to\sigma}(x, y)\,,\, x\;\bar\tau\; y\,,\, x \neq y.$$

Theorem 1 *($n \geq 2$; (Eucl.) or $n = 2$). Assume that $n = 2$ or that $(Q, +, \cdot, \leq)$ is a Euclidean field. Then in model (Q^n, τ), spacelike relatedness σ can be defined from timelike relatedness τ using 2 auxiliary variables by formula $\Psi_{\tau\to\sigma}$. Hence lightlike relatedness λ can be defined from timelike relatedness τ using 2 auxiliary variables by formula $\Psi_{\tau\to\lambda}$.*

Proof. Let us first show, for all $\bar{\mathsf{x}}, \bar{\mathsf{y}} \in Q^n$, that $\bar{\mathsf{x}}\;\sigma\;\bar{\mathsf{y}}$ holds exactly if $\Psi_{\tau\to\sigma}(\bar{\mathsf{x}}, \bar{\mathsf{y}})$ holds. To do so, assume that $\bar{\mathsf{x}}\;\bar\sigma\;\bar{\mathsf{y}}$. Then we should show that $\Psi_{\tau\to\sigma}(\bar{\mathsf{x}}, \bar{\mathsf{y}})$ does not hold. If $\bar{\mathsf{x}} = \bar{\mathsf{y}}$, then clearly $\Psi_{\tau\to\sigma}(\bar{\mathsf{x}}, \bar{\mathsf{y}})$ does not hold as it contains $\bar{\mathsf{x}} \neq \bar{\mathsf{y}}$. So assume that $\bar{\mathsf{x}} \neq \bar{\mathsf{y}}$. Let $\bar{\mathsf{z}}$ be the midpoint of line segment $\bar{\mathsf{x}}\bar{\mathsf{y}}$. Then, since $\bar{\mathsf{x}} \neq \bar{\mathsf{z}}$ and $\bar{\mathsf{z}} \neq \bar{\mathsf{y}}$ (as $\bar{\mathsf{x}} \neq \bar{\mathsf{y}}$), we should show that $\bar{\mathsf{u}}\;\tau\;\bar{\mathsf{x}}$ or $\bar{\mathsf{u}}\;\tau\;\bar{\mathsf{y}}$ holds for all $\bar{\mathsf{u}}$ for which $\bar{\mathsf{u}}\;\tau\;\bar{\mathsf{z}}$ holds. By symmetry, without loss of generality, we can assume that $\bar{\mathsf{y}}$ is in the causal future of $\bar{\mathsf{x}}$. There are two possibilities: if $\bar{\mathsf{u}}$ is in the timelike future of $\bar{\mathsf{z}}$, then $\bar{\mathsf{u}}$ is also in the timelike future of $\bar{\mathsf{x}}$ and hence $\bar{\mathsf{u}}\;\tau\;\bar{\mathsf{x}}$ holds; if $\bar{\mathsf{u}}$ is in the timelike past of $\bar{\mathsf{z}}$, then $\bar{\mathsf{u}}$ is in the timelike past of $\bar{\mathsf{y}}$ and hence $\bar{\mathsf{u}}\;\tau\;\bar{\mathsf{y}}$ holds, see Fig. 4.

To show the other direction, assume that $\bar{\mathsf{x}}\;\sigma\;\bar{\mathsf{y}}$. Without loss of generality, we can assume that $\bar{\mathsf{x}}$ and $\bar{\mathsf{y}}$ are in the same horizontal hyperplane H because there is an automorphism of (Q^n, τ, σ) that maps any two spacelike related points to horizontally related ones, cf. Proposition 1. We should show that for all $\bar{\mathsf{z}}$ which is different from $\bar{\mathsf{x}}$ and $\bar{\mathsf{y}}$, there is a $\bar{\mathsf{u}}$ timelike related to $\bar{\mathsf{z}}$, which is timelike related neither to $\bar{\mathsf{x}}$ nor to $\bar{\mathsf{y}}$. There are two cases, see Fig. 5.

- Either $\bar{\mathsf{z}}$ is not in the hyperplane H, and then there is a $\bar{\mathsf{u}} \in H$ vertically related and hence also timelike related to $\bar{\mathsf{z}}$; since $\bar{\mathsf{u}} \in H$, this $\bar{\mathsf{u}}$ is spacelike related to both $\bar{\mathsf{x}}$ and $\bar{\mathsf{y}}$;

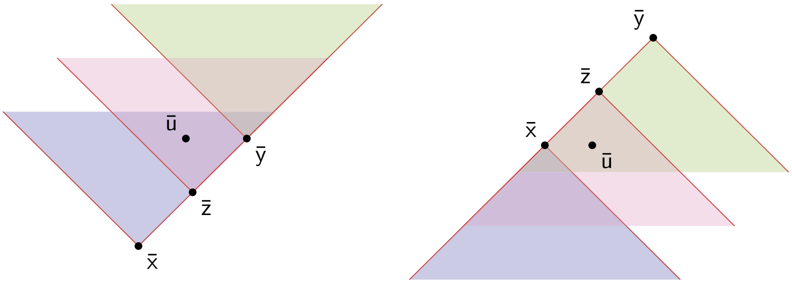

Fig. 4. Illustration for the proof of Theorem 1.

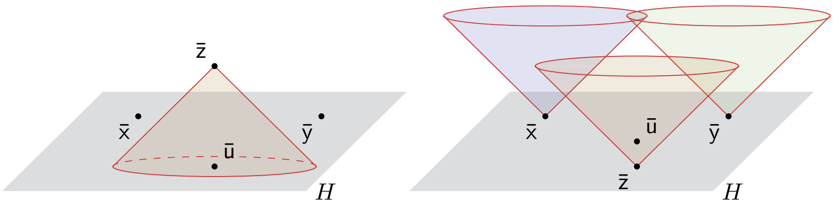

Fig. 5. Illustration for the proof of Theorem 1.

- or $\bar{z}$ is in the hyperplane H and thus spacelike related to both $\bar{x}$ and $\bar{y}$, and hence there is a close enough $\bar{u}$ which is timelike related to $\bar{z}$ but still spacelike related to $\bar{x}$ and $\bar{y}$.

The proof of the second part is straightforward since for all $\bar{x}$ and $\bar{y}$ exactly one of relations $\bar{x}\ \tau\ \bar{y}$, $\bar{x}\ \lambda\ \bar{y}$, $\bar{x}\ \sigma\ \bar{y}$ and $\bar{x} = \bar{y}$ holds. $\square$

Timelike relatedness and lightlike relatedness can be defined from spacelike relatedness by the following analogous formulas[6] (see Fig. 6):

$$\Psi_{\sigma\to\tau}(x,y) \stackrel{\text{def}}{=} x \neq y\,,\ \forall z\big(z = x \vee z = y \vee \exists u(u\ \sigma\ z\,,\ u\ \bar{\sigma}\ x\,,\ u\ \bar{\sigma}\ y)\big).$$

$$\Psi_{\sigma\to\lambda}(x,y) \stackrel{\text{def}}{=} \neg\Psi_{\sigma\to\tau}(x,y)\,,\ x\ \bar{\sigma}\ y\,,\ x \neq y.$$

Let us note here that, even though there is no symmetry between relations σ and τ unless $n = 2$, there is a nice symmetry between defining formulas $\Psi_{\sigma\to\tau}$ and $\Psi_{\tau\to\sigma}$ (as well as $\Psi_{\sigma\to\lambda}$ and $\Psi_{\tau\to\lambda}$) as they can be achieved from each other by interchanging relations σ and τ.

Theorem 2 *($n \geq 2$; (Eucl.) or $n = 2$). Assume that $n = 2$ or that $(Q, +, \cdot, \leq)$ is a Euclidean field. Then in model (Q^n, σ), timelike relatedness τ can be defined from spacelike relatedness σ using 2 auxiliary variables by formula $\Psi_{\sigma\to\tau}$. Hence lightlike relatedness λ can be defined from spacelike relatedness σ using 2 auxiliary variables by formula $\Psi_{\sigma\to\lambda}$.*

Proof. First we show, for all $\bar{x}, \bar{y} \in Q^n$, that $\bar{x}\ \tau\ \bar{y}$ holds exactly if $\Psi_{\sigma\to\tau}(\bar{x}, \bar{y})$ holds. So let us assume that $\bar{x}\ \tau\ \bar{y}$. Without loss of generality, we can assume that $\bar{x}$ and $\bar{y}$ are in the same vertical line ℓ because there is an automorphism of (Q^n, σ, τ) that maps any two timelike related points to vertically related ones, cf. Proposition 1. We should show that for all $\bar{z}$ which is different from $\bar{x}$ and $\bar{y}$, there is a $\bar{u}$ spacelike related to $\bar{z}$, which is spacelike related neither to $\bar{x}$ nor to $\bar{y}$. To find such a $\bar{u}$, let us consider the hyperplane H through $\bar{z}$ orthogonal to ℓ, see Fig. 7. If $\bar{z} \notin \ell$, let $\bar{u}$ be the intersection point of ℓ and H; and if $\bar{z} \in \ell$, let

[6] Let us note that this formula is basically the one mentioned by Malament [10, footnote 8] defining lightlike relatedness from causality relation $\bar{\sigma}$.

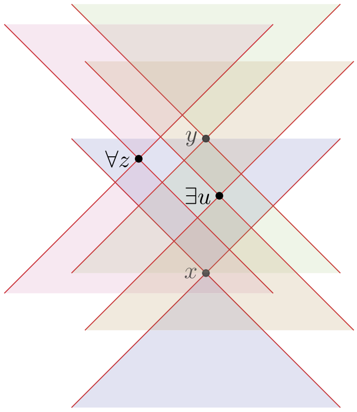

Fig. 6. Formula $\Psi_{\sigma\to\tau}(x, y)$, defining timelike relatedness τ from spacelike relatedness σ, intuitively says that x and y are distinct and outside the light cone of every point z (different from x and y) there is a point u which is outside neither the light cone of x nor that of y.

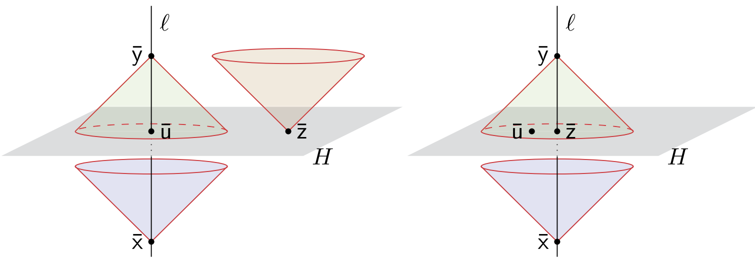

Fig. 7. Illustration for the proof of Theorem 2.

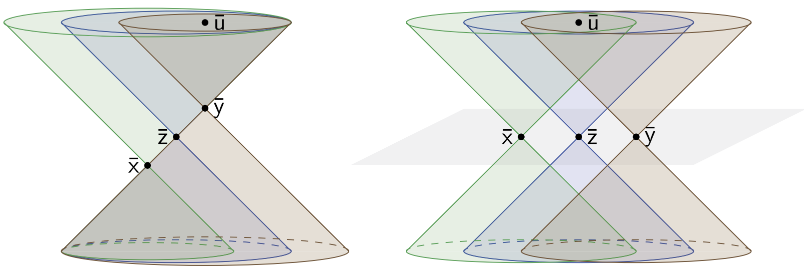

Fig. 8. Illustration for the proof of Theorem 2.

$\bar{u} \in H$ be such a point distinct from $\bar{z}$ that the distance between $\bar{u}$ and $\bar{z}$ is less than both the distance of $\bar{z}$ and $\bar{x}$ and the distance of $\bar{z}$ and $\bar{y}$. As desired, by its choice, $\bar{u}$ is spacelike related to $\bar{z}$ but not to $\bar{x}$ or $\bar{y}$.

To show the other direction, assume that $\bar{x}\ \bar{\tau}\ \bar{y}$. Then we should show that $\Psi_{\sigma\to\tau}(\bar{x}, \bar{y})$ does not hold. If $\bar{x} = \bar{y}$, then clearly $\Psi_{\sigma\to\tau}(\bar{x}, \bar{y})$ does not hold as it contains $\bar{x} \neq \bar{y}$. So assume that $\bar{x} \neq \bar{y}$ and let us choose $\bar{z}$ to be the midpoint of line segment $\bar{x}\bar{y}$, see Fig. 8. Then, since $\bar{z} \neq \bar{x}$ and $\bar{z} \neq \bar{y}$ (as $\bar{x} \neq \bar{y}$), we should show that for all $\bar{u}$, if $\bar{u}\ \bar{\sigma}\ \bar{x}$ and $\bar{u}\ \bar{\sigma}\ \bar{y}$, then $\bar{u}\ \bar{\sigma}\ \bar{z}$. There are two cases to consider: either $\bar{x}\ \lambda\ \bar{y}$ or $\bar{x}\ \sigma\ \bar{y}$ (in this second case, we can assume that $\bar{x}$ and $\bar{y}$ are horizontally related by the usual automorphism argument, cf. Proposition 1). In both cases, $\bar{u}\ \bar{\sigma}\ \bar{z}$ holds for all the points for which $\bar{u}\ \bar{\sigma}\ \bar{x}$ and $\bar{u}\ \bar{\sigma}\ \bar{y}$ because the intersection of causal cones through $\bar{x}$ and $\bar{y}$ are contained in the causal cone through $\bar{z}$.

As before, the proof of the second part is straightforward since for all $\bar{x}$ and $\bar{y}$ exactly one of relations $\bar{x}\ \tau\ \bar{y}$, $\bar{x}\ \lambda\ \bar{y}$, $\bar{x}\ \sigma\ \bar{y}$ and $\bar{x} = \bar{y}$ holds. $\square$

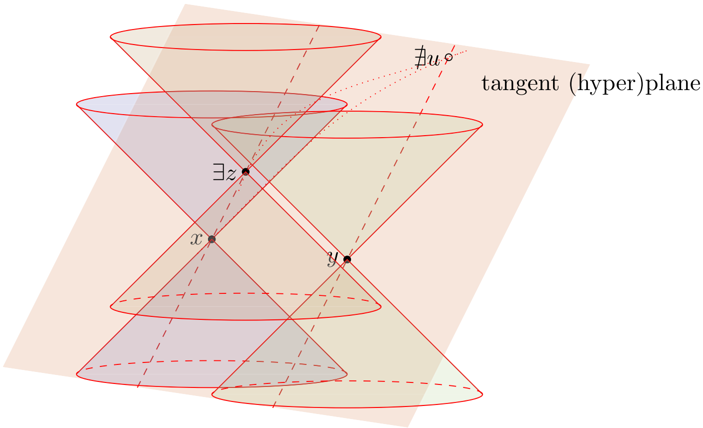

Fig. 9. Formula $\Psi_{\lambda\to\sigma}(x,y)$, defining spacelike relatedness σ from lightlike relatedness λ, intuitively says that x and y are not lightlike related, and there is a point $z \neq x$ on the light cone of x such that z is not lightlike related to y and there is no point u on the light cone of y to which both x and z are lightlike related. Since there are no non-degenerate lightlike triangles, this basically means that there is a lightlike line ℓ containing x but not y such that there is no lightlike line through y which intersects ℓ.

The following formulas are based on ideas used in [4, Appendix B]:

$$\Psi_{\lambda\to\sigma}(x,y) \overset{\text{def}}{=} x\,\bar{\lambda}\,y\,,\ \exists z\big(x\,\lambda\,z\,,\ y\,\bar{\lambda}\,z\,,\ \neg\exists u(u\,\lambda\,x\,,\ u\,\lambda\,y\,,\ u\,\lambda\,z)\big).$$

$$\Psi_{\lambda\to\tau}(x,y) \overset{\text{def}}{=} \neg\Psi_{\lambda\to\sigma}(x,y)\,,\ x\,\bar{\lambda}\,y\,,\ x \neq y.$$

Theorem 3 *((Eucl.) and $n \geq 3$). Let $n \geq 3$ and assume that ordered field $(Q,+,\cdot,\leq)$ is Euclidean. In model (Q^n,λ), spacelike relatedness σ can be defined from lightlike relatedness λ using 2 auxiliary variables by formula $\Psi_{\lambda\to\sigma}$. Hence timelike relatedness τ can be defined from lightlike relatedness λ using 2 auxiliary variables by formula $\Psi_{\lambda\to\tau}$.*

Proof. First we show, for all $\bar{\mathsf{x}},\bar{\mathsf{y}} \in Q^n$, that $\bar{\mathsf{x}}\,\sigma\,\bar{\mathsf{y}}$ holds exactly if $\Psi_{\lambda\to\sigma}(\bar{\mathsf{x}},\bar{\mathsf{y}})$ holds. Thus let us assume that $\bar{\mathsf{x}}\,\sigma\,\bar{\mathsf{y}}$. Then we should show that $\Psi_{\lambda\to\sigma}(\bar{\mathsf{x}},\bar{\mathsf{y}})$ also holds. To do so, consider the light cone $\Lambda_{\bar{\mathsf{x}}}$ through $\bar{\mathsf{x}}$ and take a tangent hyperplane H to this light cone containing $\bar{\mathsf{y}}$, such H exists because $\bar{\mathsf{x}}$ is spacelike related to $\bar{\mathsf{y}}$ and $(Q,+,\cdot,\leq)$ is a Euclidean field, see Fig. 9. Let $\bar{\mathsf{z}}$ be any point but $\bar{\mathsf{x}}$ from line $\ell := \Lambda_{\bar{\mathsf{x}}} \cap H$. Then $\bar{\mathsf{x}}\,\bar{\lambda}\,\bar{\mathsf{y}}$ holds because $\bar{\mathsf{x}}$ and $\bar{\mathsf{y}}$ are assumed to be spacelike related. Since $\bar{\mathsf{z}} \in \Lambda_{\bar{\mathsf{x}}}$ and $\bar{\mathsf{z}} \neq \bar{\mathsf{x}}$, we have that $\bar{\mathsf{x}}\,\lambda\,\bar{\mathsf{z}}$. Since it is a tangent hyperplane of a light cone, all lightlike lines are parallel in H (since H is tangent to every light cone through any point of H). Hence $\bar{\mathsf{y}}\,\bar{\lambda}\,\bar{\mathsf{z}}$ also holds since $\bar{\mathsf{y}} \notin \ell$ (as $\bar{\mathsf{y}} \notin \Lambda_{\bar{\mathsf{x}}}$). So it remains to show that there is no $\bar{\mathsf{u}}$ for which $\bar{\mathsf{u}}\,\lambda\,\bar{\mathsf{x}}$, $\bar{\mathsf{u}}\,\lambda\,\bar{\mathsf{y}}$ and $\bar{\mathsf{u}}\,\lambda\,\bar{\mathsf{z}}$. To show this, let $\bar{\mathsf{u}}$ be lightlike related to both $\bar{\mathsf{x}}$ and $\bar{\mathsf{z}}$. Then $\bar{\mathsf{u}} \in \ell$ because all lightlike triangles are degenerate[7] and $\bar{\mathsf{u}}\bar{\mathsf{x}}\bar{\mathsf{z}}$ is a lightlike triangle. Therefore, $\bar{\mathsf{u}}$ cannot be lightlike related to $\bar{\mathsf{y}}$ because all lightlike lines are parallel in H. Hence, there is no $\bar{\mathsf{u}}$ for which all of $\bar{\mathsf{u}}\,\lambda\,\bar{\mathsf{x}}$, $\bar{\mathsf{u}}\,\lambda\,\bar{\mathsf{y}}$ and $\bar{\mathsf{u}}\,\lambda\,\bar{\mathsf{z}}$ hold. Consequently, $\Psi_{\lambda\to\sigma}(\bar{\mathsf{x}},\bar{\mathsf{y}})$ holds and this is what we wanted to show.

Let us now assume that $\bar{\mathsf{x}}\,\bar{\sigma}\,\bar{\mathsf{y}}$. Then we should show that $\Psi_{\lambda\to\sigma}(\bar{\mathsf{x}},\bar{\mathsf{y}})$ does not hold, i.e., $\bar{\mathsf{x}}\,\lambda\,\bar{\mathsf{y}}$ or, for all $\bar{\mathsf{z}}$ which is lightlike related to $\bar{\mathsf{x}}$ but not lightlike related to $\bar{\mathsf{y}}$, we should be able to find a $\bar{\mathsf{u}}$ such that $\bar{\mathsf{u}}$ is lightlike related to $\bar{\mathsf{x}}$, $\bar{\mathsf{y}}$ and $\bar{\mathsf{z}}$. Assumption $\bar{\mathsf{x}}\,\bar{\sigma}\,\bar{\mathsf{y}}$ means that $\bar{\mathsf{x}} = \bar{\mathsf{y}}$, $\bar{\mathsf{x}}\,\lambda\,\bar{\mathsf{y}}$ or $\bar{\mathsf{x}}\,\tau\,\bar{\mathsf{y}}$. Since $\bar{\mathsf{x}}\,\lambda\,\bar{\mathsf{z}}$ and $\bar{\mathsf{y}}\,\bar{\lambda}\,\bar{\mathsf{z}}$ imply that $\bar{\mathsf{x}} \neq \bar{\mathsf{y}}$, the only nontrivial case to be checked is $\bar{\mathsf{x}}\,\tau\,\bar{\mathsf{y}}$.

So let us assume that $\bar{\mathsf{x}}\,\tau\,\bar{\mathsf{y}}$ and take any $\bar{\mathsf{z}}$ for which $\bar{\mathsf{z}}\,\lambda\,\bar{\mathsf{x}}$ and $\bar{\mathsf{z}}\,\bar{\lambda}\,\bar{\mathsf{y}}$ hold. Then let us consider the lightlike line containing $\bar{\mathsf{x}}$ and $\bar{\mathsf{z}}$ and the nonparallel lightlike line through $\bar{\mathsf{y}}$ in the plane determined by $\bar{\mathsf{x}}$, $\bar{\mathsf{y}}$ and $\bar{\mathsf{z}}$ (this lightlike line through $\bar{\mathsf{y}}$ exists because the plane determined by $\bar{\mathsf{x}}$, $\bar{\mathsf{y}}$ and $\bar{\mathsf{z}}$ contains a timelike line).

[7] This is so because there are tree kinds of planes: timelike planes (that contain exactly two lightlike directions), Robb planes (that contain only one lightlike directions) and spacelike planes (that contain no lightlike direction at all) and none of these planes can contain a lightlike triangle unless it is degenerate.

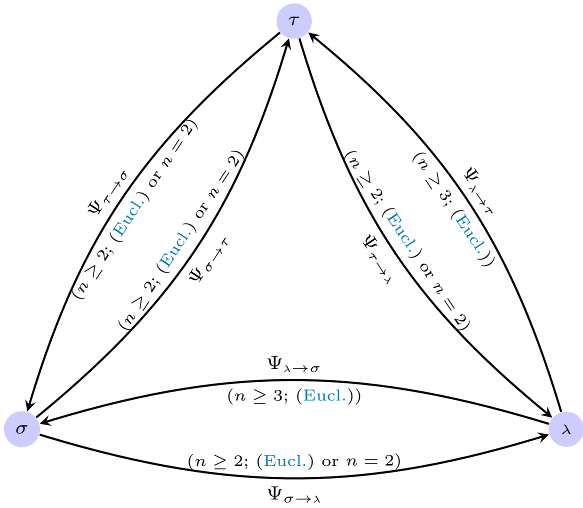

Fig. 10. Here we illustrate formulas defining timelike τ, spacelike σ, lightlike relatedness λ from each other using only 2 auxiliary variables in the corresponding space-time dimensions n.

These two lines intersect because they are coplanar nonparallel lightlike lines. Let $\bar{\mathsf{u}}$ be their intersection. By its choice, this $\bar{\mathsf{u}}$ is lightlike related to $\bar{\mathsf{x}}$, $\bar{\mathsf{u}}$ and $\bar{\mathsf{z}}$ as desired.

Again the proof of the second part is straightforward since for all $\bar{\mathsf{x}}$ and $\bar{\mathsf{y}}$ exactly one of relations $\bar{\mathsf{x}}\ \tau\ \bar{\mathsf{y}}$, $\bar{\mathsf{x}}\ \lambda\ \bar{\mathsf{y}}$, $\bar{\mathsf{x}}\ \sigma\ \bar{\mathsf{y}}$, and $\bar{\mathsf{x}} = \bar{\mathsf{y}}$ holds. $\square$

In Fig. 10, we summarize the various defining formulas of this section and the required conditions in the corresponding theorems.

Remark 2 *($n = 2$).* Over any ordered field $(Q, +, \cdot, \leq)$, if $n = 2$, neither spacelike relatedness σ nor timelike relatedness τ can be defined from lightlike relatedness λ because map $\alpha : (\mathsf{t}, \mathsf{x}) \mapsto (\mathsf{x}, \mathsf{t})$ is an automorphism of model (Q^2, λ) taking relation σ to τ.

Remark 3. The assumption that $(Q, +, \cdot, \leq)$ is Euclidean cannot be omitted from Theorem 3. To see this, let us consider the set of real numbers which can be written as $\mathsf{a} + \mathsf{b}\sqrt{2}$ for some rational numbers a and b. It is known and also easy to check that these numbers form an ordered subfield of real numbers. Map $\alpha : \mathsf{x} + \mathsf{y}\sqrt{2} \mapsto \mathsf{x} - \mathsf{y}\sqrt{2}$ preserves the addition and the multiplication in this field, and hence bijection $\hat{\alpha} : (\mathsf{p}_0, \mathsf{p}_1, \ldots, \mathsf{p}_{n-1}) \mapsto (\alpha(\mathsf{p}_0), \alpha(\mathsf{p}_1), \ldots, \alpha(\mathsf{p}_{n-1}))$ takes lines to lines, and preserves lightlike relatedness. Since $\hat{\alpha}$ interchanges timelike vector $\big(1, 1 - \sqrt{2}/2, 0 \ldots, 0\big)$ and spacelike vector $\big(1, 1 + \sqrt{2}/2, 0, \ldots, 0\big)$, neither timelike relatedness nor spacelike relatedness can be defined from lightlike relatedness over this ordered field. This counterexample can be generalized over any ordered field $(Q, +, \cdot, \leq)$ which has an automorphism α of $(Q, +, \cdot)$ which does not preserve the ordering $\leq$. This is so because in those fields there is an element $\mathsf{b} > 0$ such that $\alpha(\mathsf{b}) < 0$. If $\mathsf{b} < 1$, then $(1, 1 - \mathsf{b}, 0, \ldots 0)$ is a timelike vector whose image is spacelike, otherwise $(1, 1 - 1/\mathsf{b}, 0, \ldots, 0)$ is such a vector.

Remark 4. Even though the picturesque proof used in this paper does not work over arbitrary ordered fields, since it uses Proposition 1 (which does not hold, for example, in the field of rational numbers), Theorems 1 and 2 do hold over arbitrary ordered fields.

Theorem 4 *($n \geq 2$). Over any ordered field $(Q, +, \cdot, \leq)$, none of the relations τ, λ and σ is definable in terms of one of the others using only 1 auxiliary variable.*

Table 1
Quantifier complexity of definitions used in Section 3.

defining formula	$\Psi_{\tau\to\sigma}$	$\Psi_{\tau\to\lambda}$	$\Psi_{\sigma\to\tau}$	$\Psi_{\sigma\to\lambda}$	$\Psi_{\lambda\to\tau}$	$\Psi_{\lambda\to\sigma}$
complexity	$\forall^1\exists^1$	$\exists^1\forall^1$	$\forall^1\exists^1$	$\exists^1\forall^1$	$\forall^1\exists^1$	$\exists^1\forall^1$

Proof. By a theorem of Tarski and Givant [17], see also [2, Prop. 3.18, p. 111], a binary relation B on a set U is first-order definable from a set $\mathcal{R}$ of binary relations on U using only one auxiliary variable iff B can be built up from members of $\mathcal{R}$ and the identity relation $=$ by using the following operations: intersection $\cap$, union $\cup$, complement $^-$, relation composition ; and converse $^{-1}$. Let ρ be any of the relations τ, λ and σ. To show that none of the other relations is definable from ρ, we are going to show that set

$$\mathcal{S} = \left\{\emptyset,\ =,\ \neq,\ \rho,\ \bar{\rho},\ \bar{\rho}\cap\neq,\ \rho\cup=,\ Q^n\times Q^n\right\}$$

is closed under the above relation operations. $\mathcal{S}$ is clearly closed under complement by De Morgan's law and it is easy to check that $\mathcal{S}$ is closed under intersection and union, cf. Fig. 2. Hence $\mathcal{S}$ is a *field of sets*[8] with atoms $=$, ρ and $\bar{\rho}_{\neq}$ (which is $\bar{\rho}\cap\neq$). Since identity $(P\cup Q)\,;R \equiv (P\,;R)\cup(Q\,;R)$ holds for all binary relations P, Q and R, it is enough to show that the various compositions of atoms $=$, ρ and $\bar{\rho}_{\neq}$ are in $\mathcal{S}$, which holds because $\rho\,;\rho$ and $\bar{\rho}_{\neq}\,;\bar{\rho}_{\neq}$ are the universal relation $Q^n\times Q^n$, compositions $\bar{\rho}_{\neq}\,;\rho$ and $\rho\,;\bar{\rho}_{\neq}$ are equal to the relation $\neq$, and $=$ is an identity element with respect to the relation composition. Since all the relations in $\mathcal{S}$ are symmetric, they are equal to their converses. Therefore, all the binary relations first-order definable from ρ using only one auxiliary variable are in $\mathcal{S}$. This completes the proof of the theorem. □

4. Quantifier complexity

Let us now investigate the quantifier complexity of the possible definitions. To do so, let us recall that a formula is called ***universal*** iff it is of form $\forall x_1\ldots\forall x_k\psi$ for some quantifier free formula ψ and variables $x_1,\ldots,x_k$. A formula is ***existential*** in the completely analogous case when $\forall$ is replaced by $\exists$. A formula is called ***universal-existential*** iff it is of form $\forall y_1\ldots\forall y_m\exists x_1\ldots\exists x_k\psi$ for some quantifier free formula ψ and variables $x_1,\ldots,x_k,y_1\ldots,y_m$. The definition of an ***existential-universal*** formula can be obtained from this by interchanging the universal and existential quantifiers. In the superscript, we indicate the number of quantifies, and we use $*$ in case this number is not specified, e.g., by $\forall^1\exists^1$-formulas we mean universal-existential formulas containing exactly one universal and one existential quantifiers, and by $\forall^*\exists^*$-formulas we mean all universal-existential formulas.

It is straightforward to show that the defining formulas $\Psi_{\tau\to\sigma}$, $\Psi_{\sigma\to\tau}$ and $\Psi_{\lambda\to\tau}$ are logically equivalent to $\forall^1\exists^1$-formulas and $\Psi_{\tau\to\lambda}$, $\Psi_{\sigma\to\lambda}$ and $\Psi_{\lambda\to\sigma}$ are logically equivalent to $\exists^1\forall^1$-formulas, see Table 1. So these formulas are also quite simple in terms of quantifiers.

It is a natural question whether it is possible to find even simpler defining formulas in terms of quantifier complexity. Now we are going to investigate this question looking for existential and universal definitions. To make our formulas more concise, we are going to use the following abbreviation for certain binary relations ρ and δ:

$$x\ \rho\delta\ yz \stackrel{\text{def}}{\iff} x\ \rho\ y\,,\ x\ \delta\ z.$$

We will also use the natural generalization of this abbreviation for more than two binary relations.

In Theorem 5 below, we are going to show that the following $\exists^4$-formula $\mathcal{E}_{\tau\to\sigma}$ defines spacelike relatedness σ from timelike relatedness τ in (Q^n,τ):

[8] That is a subalgebra of the power set Boolean algebra of $Q^n\times Q^n$.

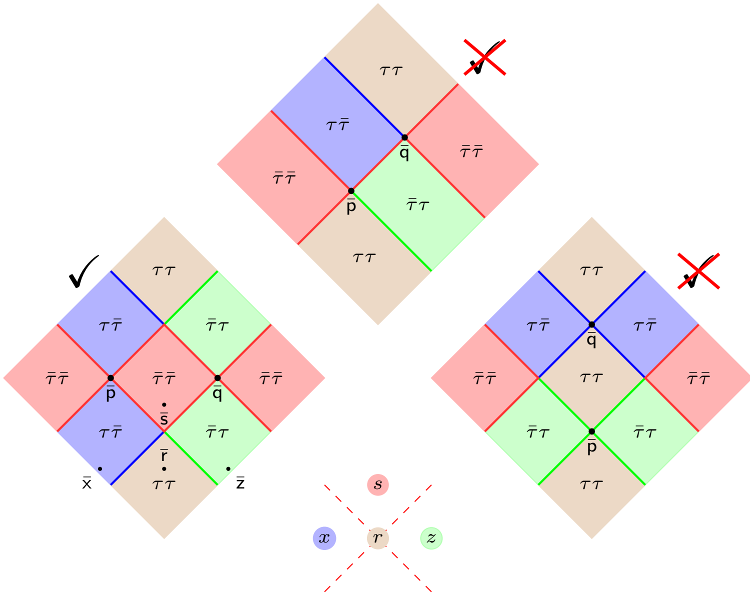

Fig. 11. Here, we illustrate the proof of that $\mathcal{E}_{\tau\to\sigma}(\bar{\mathsf{p}},\bar{\mathsf{q}})$ holds exactly if $\bar{\mathsf{p}}\ \sigma\ \bar{\mathsf{q}}$ holds. (For colored versions of the figure(s), the reader is referred to the web version of this article.)

$$\mathcal{E}_{\tau\to\sigma}(p,q) \stackrel{\mathsf{def}}{=} \exists r\exists x\exists s\exists z(r\ \tau\tau\ pq\,,\ x\ \tau\bar\tau\ pq\,,\ s\ \bar\tau\bar\tau\ pq\,,\ z\ \bar\tau\tau\ pq\,,\ r\ \bar\tau\tau\bar\tau\ xsz),$$

and hence

$$\mathcal{U}_{\tau\to\lambda}(p,q) \stackrel{\mathsf{def}}{=} \neg\mathcal{E}_{\tau\to\sigma}(p,q)\,,\ p\neq q\,,\ p\ \bar\tau\ q$$

gives a universal definition of lightlike relatedness from τ.

Theorem 5 *($n \geq 2$; (Eucl.) or $n = 2$). Assume that $n = 2$ or that $(Q, +, \cdot, \leq)$ is a Euclidean field, then:*

1. *Spacelike relatedness σ can be defined existentially from timelike relatedness τ by $\exists^4$-formula $\mathcal{E}_{\tau\to\sigma}$ in (Q^n, τ).*
2. *Lightlike relatedness λ can be defined universally from timelike relatedness τ by $\forall^4$-formula $\mathcal{U}_{\tau\to\lambda}$ in (Q^n, τ).*

Proof. First we are going to show that $\bar{\mathsf{p}}\ \sigma\ \bar{\mathsf{q}}$ holds exactly if $\mathcal{E}_{\tau\to\sigma}(\bar{\mathsf{p}},\bar{\mathsf{q}})$ holds. If $\bar{\mathsf{p}}\ \sigma\ \bar{\mathsf{q}}$, then we can assume, without loosing generality, that $\bar{\mathsf{p}}$ and $\bar{\mathsf{q}}$ are horizontally related, cf. Proposition 1. In this case, it is easy to verify that there are (even coplanar) points $\bar{\mathsf{r}},\bar{\mathsf{x}},\bar{\mathsf{s}},\bar{\mathsf{z}} \in Q^n$ for which relations $\bar{\mathsf{r}}\ \tau\tau\ \bar{\mathsf{p}}\bar{\mathsf{q}}$, $\bar{\mathsf{x}}\ \tau\bar\tau\ \bar{\mathsf{p}}\bar{\mathsf{q}}$, $\bar{\mathsf{s}}\ \bar\tau\bar\tau\ \bar{\mathsf{p}}\bar{\mathsf{q}}$, $\bar{\mathsf{z}}\ \bar\tau\tau\ \bar{\mathsf{p}}\bar{\mathsf{q}}$, $\bar{\mathsf{r}}\ \tau\ \bar{\mathsf{s}}$ and $\bar{\mathsf{r}}\ \bar\tau\bar\tau\ \bar{\mathsf{x}}\bar{\mathsf{z}}$ hold; and hence $\mathcal{E}_{\tau\to\sigma}(\bar{\mathsf{p}},\bar{\mathsf{q}})$ also holds, see Fig. 11, where the regions are colored and labeled based on how the points from there are respectively related to $\bar{\mathsf{p}}$ and $\bar{\mathsf{q}}$. So, according to Fig. 11, $\bar{\mathsf{r}}\ \tau\tau\ \bar{\mathsf{p}}\bar{\mathsf{q}}$ holds iff $\bar{\mathsf{r}}$ is in a brown $\tau\tau$ region, $\bar{\mathsf{x}}\ \tau\bar\tau\ \bar{\mathsf{p}}\bar{\mathsf{q}}$ holds iff $\bar{\mathsf{x}}$ is in a blue $\tau\bar\tau$ region, $\bar{\mathsf{s}}\ \bar\tau\bar\tau\ \bar{\mathsf{p}}\bar{\mathsf{q}}$ holds iff $\bar{\mathsf{s}}$ is in a red $\bar\tau\bar\tau$ region, and $\bar{\mathsf{z}}\ \bar\tau\tau\ \bar{\mathsf{p}}\bar{\mathsf{q}}$ holds iff $\bar{\mathsf{z}}$ in a green $\bar\tau\tau$ region.

To prove the other direction, let us assume that $\bar{\mathsf{p}}\ \sigma\ \bar{\mathsf{q}}$ does not hold and prove that $\mathcal{E}_{\tau\to\sigma}(\bar{\mathsf{p}},\bar{\mathsf{q}})$ does not hold. Then there are three cases to consider: either $\bar{\mathsf{p}} = \bar{\mathsf{q}}$ or $\bar{\mathsf{p}}\ \lambda\ \bar{\mathsf{q}}$ or $\bar{\mathsf{p}}\ \tau\ \bar{\mathsf{q}}$.

If $\bar{\mathsf{p}}\ \tau\ \bar{\mathsf{q}}$, then, without loosing generality, we can assume that $\bar{\mathsf{q}}$ is in the timelike future of $\bar{\mathsf{p}}$ and that $\bar{\mathsf{p}}$ and $\bar{\mathsf{q}}$ are vertically related by Proposition 1. In this case, we are going to show that there are no appropriate points $\bar{\mathsf{r}},\bar{\mathsf{x}},\bar{\mathsf{s}},\bar{\mathsf{z}} \in Q^n$. Even though Fig. 11 is two-dimensional, it also illustrates the higher-dimensional cases well because of rotational symmetry with respect to the time-axis. So we are going to use Fig. 11 to refer the regions having appropriate relations with respect to $\bar{\mathsf{p}}$ and $\bar{\mathsf{q}}$.

Relation $\mathcal{E}_{\tau\to\sigma}(\bar{\mathsf{p}},\bar{\mathsf{q}})$ requires $\bar{\mathsf{r}}$ to be in one of the three brown $\tau\tau$ regions. Now we are going to show that choosing $\bar{\mathsf{r}}$ from any of these three regions leads to contradiction.

- If $\bar{\mathsf{r}}$ is in the bottom brown $\tau\tau$ region, then there is no $\bar{\mathsf{x}}$ from a blue $\tau\bar{\tau}$ region, which is $\bar{\tau}$-related to $\bar{\mathsf{r}}$. This is so because of the followings. Such an $\bar{\mathsf{x}}$ should be in the timelike future of $\bar{\mathsf{p}}$ because, if it was in the timelike past of $\bar{\mathsf{p}}$, then (by transitivity) it would also be in the timelike past of $\bar{\mathsf{q}}$ contradicting that $\bar{\mathsf{x}}\ \bar{\tau}\ \bar{\mathsf{q}}$. Thus, since $\bar{\mathsf{p}}$ is in the timelike future of $\bar{\mathsf{r}}$, by the transitivity, $\bar{\mathsf{x}}$ should also be in the timelike future of $\bar{\mathsf{r}}$; and hence relation $\bar{\mathsf{r}}\ \bar{\tau}\ \bar{\mathsf{x}}$ cannot hold.
- If $\bar{\mathsf{r}}$ is in the middle brown $\tau\tau$ region, then there is no $\bar{\mathsf{s}}$ from a red $\bar{\tau}\bar{\tau}$ region which is τ-related to $\bar{\mathsf{r}}$. This is so because, if $\bar{\mathsf{s}}$ is timelike related to $\bar{\mathsf{r}}$, then it is also timelike related to $\bar{\mathsf{p}}$ (if $\bar{\mathsf{s}}$ is in the future of $\bar{\mathsf{r}}$) or timelike related to $\bar{\mathsf{q}}$ (if $\bar{\mathsf{s}}$ is in the past of $\bar{\mathsf{r}}$). Thus $\bar{\mathsf{s}}$ cannot be $\bar{\tau}$-related to both $\bar{\mathsf{p}}$ and $\bar{\mathsf{q}}$; and hence it cannot be from a red $\bar{\tau}\bar{\tau}$ region.
- If $\bar{\mathsf{r}}$ is in the upper brown $\tau\tau$ region, then there is no $\bar{\mathsf{z}}$ from a green $\bar{\tau}\tau$ region, which is $\bar{\tau}$-related to $\bar{\mathsf{r}}$ for a completely analogous reason as in the first case.

Hence there is no appropriate $\bar{\mathsf{r}}$ required by formula $\mathcal{E}_{\tau\to\sigma}$. Therefore, $\mathcal{E}_{\tau\to\sigma}(\bar{\mathsf{p}},\bar{\mathsf{q}})$ cannot hold if $\bar{\mathsf{p}}$ and $\bar{\mathsf{q}}$ are timelike related.

If $\bar{\mathsf{p}}\ \lambda\ \bar{\mathsf{q}}$, then basically the same argument works but it is simpler because there is no middle brown $\tau\tau$ region, cf. Fig. 11. Here, instead of the transitivity of the timelike past and future relations, we should use the fact that the timelike past of points from the causal past of point $\bar{\mathsf{q}}$ is in the timelike past of $\bar{\mathsf{q}}$, and the analogous fact that we get from this one if we replace past with future and $\bar{\mathsf{q}}$ with $\bar{\mathsf{p}}$. If $\bar{\mathsf{p}}=\bar{\mathsf{q}}$, then the situation is even simpler because, then there are no green $\tau\bar{\tau}$ and blue $\bar{\tau}\tau$ regions at all. So, if $\bar{\mathsf{p}}\ \sigma\ \bar{\mathsf{q}}$ does not hold, then $\mathcal{E}_{\tau\to\sigma}(\bar{\mathsf{p}},\bar{\mathsf{q}})$ does not hold either. This completes the proof of this direction and hence the proof of Item (1).

Item (2) follows from Item (1) and the fact that exactly one of relations $\bar{\mathsf{p}}\ \tau\ \bar{\mathsf{q}}$, $\bar{\mathsf{p}}\ \lambda\ \bar{\mathsf{q}}$, $\bar{\mathsf{p}}\ \sigma\ \bar{\mathsf{q}}$, and $\bar{\mathsf{p}}=\bar{\mathsf{q}}$ holds for every $\bar{\mathsf{p}},\bar{\mathsf{q}}\in Q^n$. □

Since map $\alpha:(\mathsf{t},\mathsf{x})\mapsto(\mathsf{x},\mathsf{t})$ is an isomorphism between structures (Q^2,τ,σ) and (Q^2,σ,τ), existential formula

$$\mathcal{E}_{\sigma\to\tau}(p,q)\overset{\mathsf{def}}{=}\exists r\exists x\exists s\exists z(r\ \sigma\sigma\ pq\,,\ x\ \sigma\bar{\sigma}\ pq\,,\ s\ \bar{\sigma}\bar{\sigma}\ pq\,,\ z\ \bar{\sigma}\sigma\ pq\,,\ r\ \bar{\sigma}\sigma\bar{\sigma}\ xsz)$$

defines timelike relatedness τ from spacelike relatedness σ in (Q^2,σ), and hence,

$$\mathcal{U}_{\sigma\to\lambda}(p,q)\overset{\mathsf{def}}{=}\neg\mathcal{E}_{\sigma\to\tau}(p,q)\,,\ p\neq q\,,\ p\ \bar{\sigma}\ q$$

gives a universal definition of lightlike relatedness from σ in (Q^2,σ). In other words, following is an immediate corollary of Theorem 5.

Corollary 3 *($n=2$). Let $(Q,+,\cdot,\leq)$ be an arbitrary ordered field, then:*

1. *Timelike relatedness τ can be defined existentially from spacelike relatedness σ by $\exists^4$-formula $\mathcal{E}_{\sigma\to\tau}$ in (Q^2,σ).*
2. *Lightlike relatedness λ can be defined universally from spacelike relatedness σ by $\forall^4$-formula $\mathcal{U}_{\sigma\to\lambda}$ in (Q^2,σ).* □

Remark 5 *($n\geq 3$).* The assumption $n=2$ cannot be omitted from Corollary 3, i.e., $\exists^4$-formula $\mathcal{E}_{\sigma\to\tau}$ does not define τ in (Q^n,σ) if $n\geq 3$. Spacelike related points $\bar{\mathsf{p}}$ and $\bar{\mathsf{q}}$ satisfying $\mathcal{E}_{\sigma\to\tau}(p,q)$ can easily be found searching them in horizontal slices of (Q^n,σ) looked from above, cf. Fig. 12. That $\mathcal{E}_{\sigma\to\tau}$ can also be

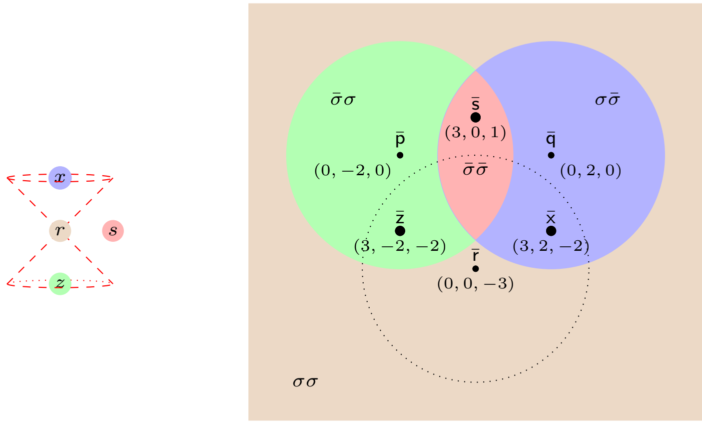

Fig. 12. This figure illustrates why $\exists^4$-formula $\mathcal{E}_{\sigma\to\tau}$ does not define timelike relatedness τ in (Q^n, σ) if $n \geq 3$. It shows a horizontal slice of (Q^3, σ) viewed from above together with the various regions of this plane colored and labeled how the points from there are related to the horizontally related points $\bar{\mathsf{p}} = (0, -2, 0)$ and $\bar{\mathsf{q}} = (0, 2, 0)$ below this plane. Points $\bar{\mathsf{x}} = (3, 2, -2)$, $\bar{\mathsf{s}} = (3, 0, 1)$ and $\bar{\mathsf{z}} = (3, -2, -2)$ are in this horizontal plane, points $\bar{\mathsf{p}}$, $\bar{\mathsf{q}}$ and $\bar{\mathsf{r}} = (0, 0, -3)$ are below in a parallel plane. Checking that these points show that spacelike related points $\bar{\mathsf{p}}$ and $\bar{\mathsf{q}}$ satisfy $\mathcal{E}_{\sigma\to\tau}(p, q)$ is straightforward both by calculation and by the figure using the symmetries of the construction.

satisfied by lightlike related points in (Q^n, σ) if $n \geq 3$ can be shown by checking the following example: $\bar{\mathsf{p}} = (-2, -2, 0)$, $\bar{\mathsf{s}} = (0, 0, 0)$, $\bar{\mathsf{q}} = (2, 2, 0)$, $\bar{\mathsf{x}} = (-2, 0, 0)$, $\bar{\mathsf{z}} = (2, 0, 0)$ and $\bar{\mathsf{r}} = (0, 0, 1)$. Hence, $\forall^4$-formula $\mathcal{U}_{\sigma\to\lambda}$ does not define λ if $n \geq 3$ because the relation defined by it does not contain lightlike related point pair $\bar{\mathsf{p}} = (-2, -2, 0)$ and $\bar{\mathsf{q}} = (2, 2, 0)$.

Using p and q as free variables, up to logical equivalence, any $\exists^k$-existential definition of a binary relation from a binary relation ρ is of the following form

$$\exists z_1 \exists z_2 \ldots \exists z_k \; B(p, q, z_1, \ldots, z_k),$$

where B is some Boolean combination of relations ρ and $=$ between variables $z_1, \ldots, z_k$, p and q. Because B can be written as a disjunctive normal form and $\exists z(\varphi \vee \psi)$ is logically equivalent to $(\exists z\varphi) \vee (\exists z\psi)$, every such definition is equivalent to a disjunction $\varphi_1 \vee \ldots \vee \varphi_m$ of existential formulas φ_i each of which requires the existence of k-many points $z_1, \ldots, z_k$ and some relations of ρ, $\bar{\rho}$, $\neq$, and $=$ between $z_1, \ldots, z_k$, p and q.

Without loosing generality, we can assume that none of the requirements between two variables in φ_i is the equality because, in that case, either the same can be defined by using less variables or $\varphi(p, q)$ is equivalent to $p = q$. We can also assume that these requirements are *non-self-contradicting* (i.e., we do not require a direct contradiction between variables $z_1, \ldots, z_k$, p and q, e.g., if we require $z_1 \; \rho \; p$, then we do not also require $z_1 \; \bar{\rho} \; p$) because then φ_i clearly defines the empty relation. This means that we can assume that between any two variables either nothing or one of relations ρ, $\bar{\rho}$, $\neq$, $\rho_{\neq}$, and $\bar{\rho}_{\neq}$ is required.

So, to understand relations definable by $\exists^k$-existential formulas, it is enough to understand the relations which are definable by an $\exists^k$-existential formula of the form $\exists z_1 \exists z_2 \ldots \exists z_k \;\; C(p, q, z_1, \ldots, z_k)$, where $C(p, q, z_1, \ldots, z_k)$ is a non-self-contradicting conjunction of relations ρ, $\bar{\rho}$, $\neq$ between variables $z_1, \ldots, z_k$, p and q (since every nonempty definable relation is the union of ones definable by such formulas). Let us call such formulas ***basic*** $\exists^k$***-formulas***.[9] The relation defined by a basic $\exists^k$-formula $\exists z_1 \exists z_2 \ldots \exists z_k \;\; C(p, q) \wedge D(p, q, z_1, \ldots, z_k)$ is the intersection of relation $C(p, q)$ and the relation defined by basic $\exists^k$-formula $\exists z_1 \exists z_2 \ldots \exists z_k \; D(p, q, z_1, \ldots, z_k)$. We are going to call a basic $\exists^k$-formula ***non-requiring***

[9] Note that everything which is definable by a basic $\exists^k$ formula ψ can easily be defined by a basic $\exists^{k+1}$ formula, e.g., $\exists v\psi$ defines the same if variable v does not occur in ψ at all.

if it does not require anything between its free variables. So every relation definable by a basic $\exists^k$-formula can be constructed using intersections from ρ, $\bar{\rho}$, $\neq$ and a relation definable by a non-requiring basic $\exists^k$-formula.

It is the most difficult to satisfy those basic $\exists^k$-formulas, which require either ρ or $\bar{\rho}_{\neq}$ between any two of variables $z_1, \ldots, z_k$, p and q. We are going to call such basic $\exists^k$-formulas ***fastidious***. By a non-requiring fastidious basic $\exists^k$-formula, we mean one that requires either ρ or $\bar{\rho}_{\neq}$ between any two variables except p and q, and there is no requirement between p and q. To gain a picturesque understanding of their satisfiability, we introduce mappings of some edge-labeled graphs to Q^n.

Let $G = (V, E)$ be a simple graph whose edges are labeled by some labeling l that maps the edges of G to a set of symmetric binary relations on Q^n. By an ***embedding*** *of* (G, l) *to* Q^n, we understand a one-to-one map $f : V \to Q^n$ such that, for all vertices $x, y \in V$, if edge $(x, y) \in E$ is labeled by relation ρ, then $f(x)\ \rho\ f(y)$ holds. We say that edge-labeled graph (G, l) is ***embeddable*** *to* Q^n if there is an embedding of (G, l) to Q^n.

To every basic $\exists^k$-formula φ, we can associate an edge-labeled simple graph whose vertices are the variables of φ, two variables x and y are connected by an edge if φ requires something between variables x and y, and this edge (x, y) is labeled by the binary relation that φ requires between x and y. A basic $\exists^k$-formula is satisfiable in model (Q^n, τ) iff the corresponding edge-labeled graph is embeddable to Q^n.

Let us start with some simple observations about these graph embeddings. A restriction of an embedding to a subset $S \subseteq V$ of vertices is an embedding of the subgraph of G induced by S. This gives us the following:

O1 If an edge-labeled graph is embeddable to Q^n, then so are its induced subgraphs.

Since it is more difficult to satisfy stronger requirements, we have:

O2 If $\rho \subseteq \delta$ are symmetric binary relations on Q^n and edge-labeled graph (G, l) is embeddable to Q^n, then so is the one which we get from (G, l) by replacing labels ρ with δ.

Since $(Q^2, \tau, \lambda, \sigma)$ can be embedded as a substructure to $(Q^n, \tau, \lambda, \sigma)$ for all $n \geq 2$, we have that:

O3 If an edge-labeled graph labeled by relations τ, $\bar{\tau}_{\neq}$, λ, σ is embeddable to Q^2, then so is to Q^n for all $n \geq 2$.

Because map $\alpha : (\mathsf{t}, \mathsf{x}) \mapsto (\mathsf{x}, \mathsf{t})$ preserves relation λ but interchanges τ and σ in $(Q^2, \tau, \lambda, \sigma)$, we have that:

O4 If (G, l) labeled by τ, λ and σ is embeddable to Q^2, then so is the one that we get from (G, l) by interchanging labels τ and σ.

Now using these observations, we are going to show some properties of relations definable from τ or σ by a non-requiring $\exists^2$-formula.

Lemma 1 *($n \geq 2$). Let $(Q, +, \cdot, \leq)$ be an arbitrary ordered field, then:*

1. *Every relation defined by a non-requiring basic $\exists^2$-formula contains timelike, spacelike and lightlike related pairs of points both in (Q^n, τ) and (Q^n, σ).*
2. *If $n = 2$ or $(Q, +, \cdot, \leq)$ is a Euclidean field, then every non-requiring basic $\exists^2$-formula defines either $\neq$ or the universal relation $Q^n \times Q^n$ both in (Q^n, τ) and (Q^n, σ).*

Proof. The proofs of both items in the cases (Q^n, τ) and (Q^n, σ) are completely analogous by symmetries and because every construction used in the proof is two-dimensional. Therefore, we only show the statement

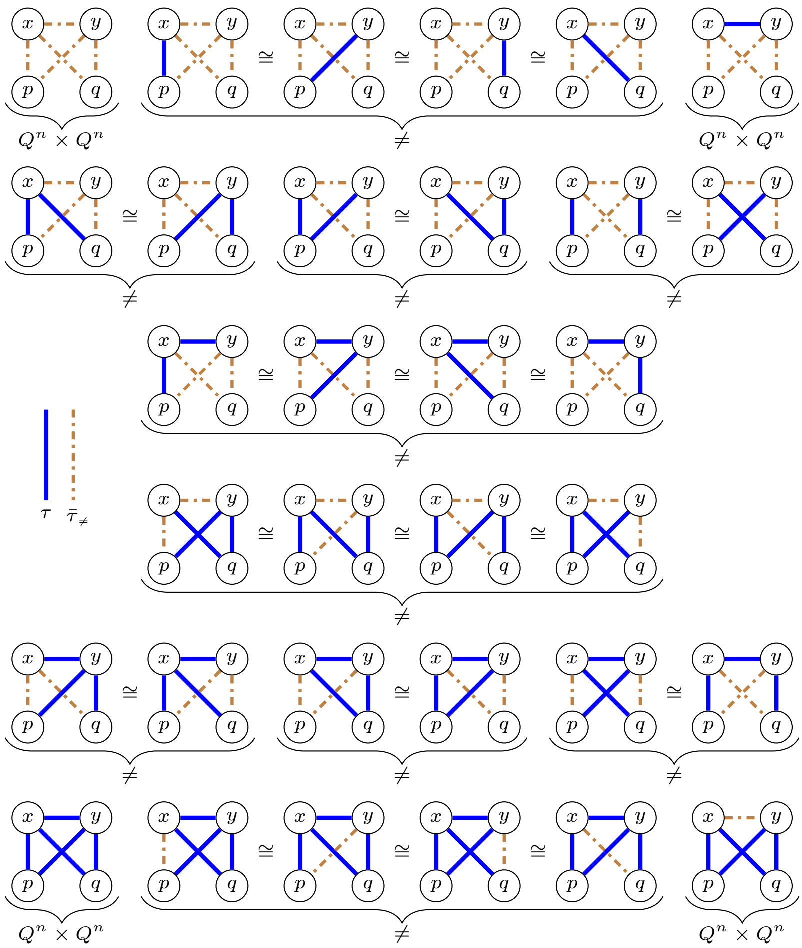

Fig. 13. The figure illustrates the edge-labeled graphs of all the non-requiring fastidious $\exists^2$-formulas and the relations what the corresponding formulas define if $n = 2$ or $(Q, +, \cdot, \leq)$ is a Euclidean field.

in (Q^n, τ). It is easy to check that the non-requiring fastidious basic $\exists^2$-formulas correspond exactly to the edge-labeled graphs listed in Fig. 13.

Fig. 14 shows how the edge-labeled graphs corresponding to non-requiring fastidious basic $\exists^2$-formulas can be embedded to Q^2 such that the $\bar{\tau}_{\neq}$ edges are mapped to spacelike related points and p and q are mapped to lightlike related if these formulas have less than three τ-requirements. The constructions illustrated by Fig. 14 can easily be modified by rising/lowering $\bar{\mathsf{q}}$ a little bit to turn the relation between $\bar{\mathsf{p}}$ and $\bar{\mathsf{q}}$ into timelike/spacelike one without changing the timelike and spacelike relatedness between any other pair of points from $\bar{\mathsf{x}}$, $\bar{\mathsf{y}}$, $\bar{\mathsf{p}}$ and $\bar{\mathsf{q}}$. In other words, we get an edge-labeled graph embeddable to Q^2 from any of the graphs from the top half of Fig. 13 even if we strengthen the $\bar{\tau}_{\neq}$-labels to σ and connect the vertices p and q labeling them by any of relations τ, σ and λ. So by observation O4, the same holds for the edge-labeled graphs from the bottom half of Fig. 13. By O3, this implies the statement of Item 1 for relations definable by a non-requiring fastidious basic $\exists^2$-formula. From this, Item 1 of the lemma follows because a non-requiring basic $\exists^2$-formula can only define something larger than its non-requiring fastidious strengthenings.

Now Item 2 follows from Item 1 by Corollary 1 and the fact that no relation between $\neq$ and $Q^n \times Q^n$ is definable from τ (as no relation between $\emptyset$ and $=$ is definable from τ). $\square$

Theorem 6 *($n \geq 2$). Let $(Q, +, \cdot, \leq)$ be an arbitrary ordered field, then:*

1. *Neither spacelike nor lightlike relatedness can be defined by an $\exists^2$-formula or a $\forall^2$-formula in (Q^n, τ).*

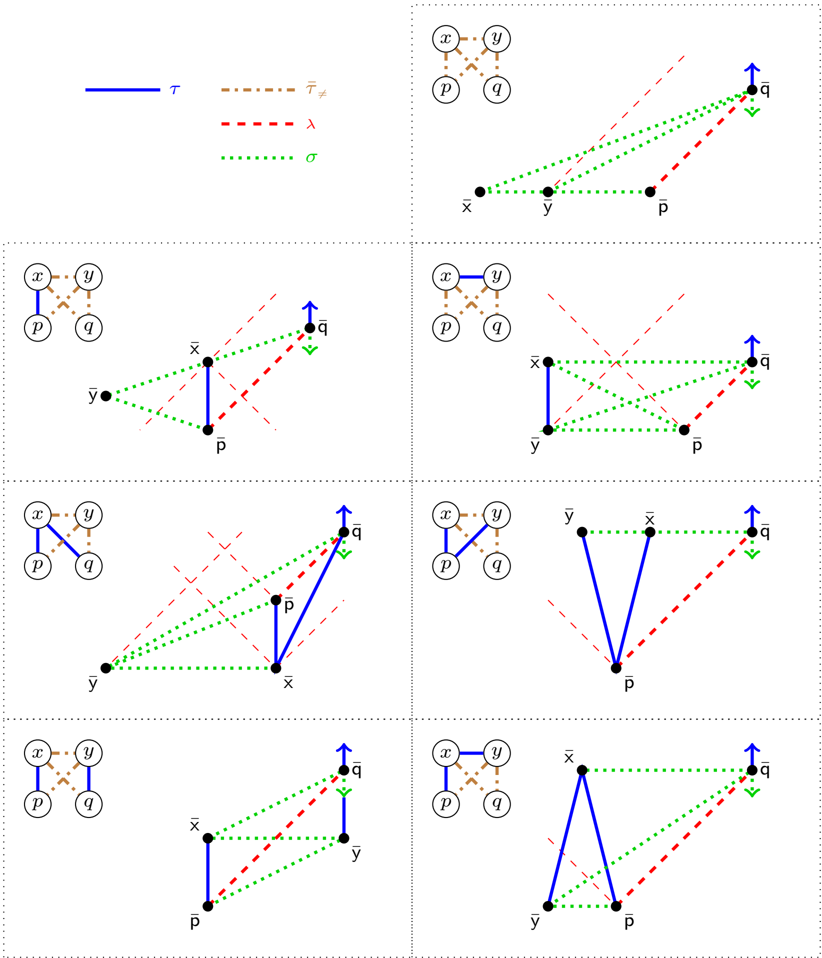

Fig. 14. The figure illustrates how one can satisfy any basic non-requiring fastidious $\exists^2$-formula φ in (Q^2, τ) mapping its free variables to arbitrary two distinct points.

2. *Neither timelike nor lightlike relatedness can be defined by an $\exists^2$-formula or a $\forall^2$-formula in (Q^n, σ).* □

Proof. In both cases, it is enough to show the nonexistence of existential definitions because that implies the nonexistence of universal ones. This is so because exactly one of relations τ, λ, σ and $=$ holds. Hence, for example, the negation of a universal definition of σ defined from τ can be turned into an existential definition of λ from τ by intersecting it with $\neq$ and $\bar{\tau}$.

By Item 1 of Lemma 1, a relation defined from τ by a non-requiring basic $\exists^2$-formula contains timelike, lightlike and spacelike pairs of points alike. So every relation which is definable from τ by a basic $\exists^2$-formula contains a spacelike related point pair exactly if it contains a lightlike related one because every such relation is the intersections of a relation definable from τ by a non-requiring basic $\exists^2$-formula and some of relations τ, $\bar{\tau}$, $\neq$ and $=$. Since every relation definable by an $\exists^2$-formula is the union of relations definable by basic $\exists^2$-formulas, every relation which is definable from τ by a $\exists^2$-formula contains a spacelike related point pair exactly if it contains a lightlike related one. Consequently, neither σ nor λ is definable from τ by an $\exists^2$-formula.

The proof of that neither τ nor λ is definable from σ by an $\exists^2$-formula is completely analogous. □

If the $n = 2$ or $(Q, +, \cdot, \leq)$ is a Euclidean field, then we can prove more:

Theorem 7 *($n \geq 2$; (Eucl.) or $n = 2$). Assume that $n = 2$ or that $(Q, +, \cdot, \leq)$ is a Euclidean field. Then we have the followings:*

1. *If a binary relation R is definable by an $\exists^2$-formula or a $\forall^2$-formula in (Q^n, τ), then R is also definable by a quantifier free formula; in other words, R has to be one of the 8 relations built up from τ and $=$ by Boolean operations, cf. Fig. 2.*
2. *If a binary relation R is definable by an $\exists^2$-formula or a $\forall^2$-formula in (Q^n, σ), then R is also definable by a quantifier free formula; in other words, R has to be one of the 8 relations built up from σ and $=$ by Boolean operations, cf. Fig. 2.*

Proof. Because the negation of a quantifier free formula is also quantifier free and every $\forall^2$-formula is equivalent to the negation of a $\exists^2$-formula, it is enough to show the existential case of both items. We are going to prove the two items simultaneously. Let ρ be one of the relations τ and σ.

Since relations definable by $\exists^2$-formulas are unions of ones definable by basic $\exists^2$-formulas, it is enough to see that the statement holds for basic $\exists^2$-formulas. Since every relation definable by basic $\exists^2$-formula φ is the intersection of a relation definable by a non-requiring basic $\exists^2$-formula and the Boolean-definable relation that φ requires between its free variables, it is enough to see that every relation definable by a non-requiring basic $\exists^2$-formula is also Boolean definable from ρ and $=$. This follows by Item 2 of Lemma 1, which completes the proof of the theorem. $\square$

Using this colorful graph embedding perspective, we are going to show that the following $\exists^3$-formula defines σ from τ in (Q^n, τ):

$$\hat{\mathcal{E}}_{\tau\to\sigma}(p, q) \stackrel{\mathsf{def}}{=} \exists x \exists y \exists z (x\ \tau\tau\bar{\tau}_{\neq}\bar{\tau}_{\neq}\ pyqz\,,\ y\ \bar{\tau}_{\neq}\bar{\tau}_{\neq}\tau\ pqz\,,\ z\ \bar{\tau}_{\neq}\tau\ pq).$$

From this, we will see that the following $\forall^3$-formula defines λ from τ in (Q^n, τ):

$$\hat{\mathcal{U}}_{\tau\to\lambda}(p, q) \stackrel{\mathsf{def}}{=} \neg\hat{\mathcal{E}}_{\tau\to\sigma}(p, q)\,,\ p \neq q\,,\ p\ \bar{\tau}\ q.$$

Theorem 8 *($n \geq 2$; (Eucl.) or $n = 2$). Assume that $n = 2$ or that $(Q, +, \cdot, \leq)$ is a Euclidean field, then:*

1. *Spacelike relatedness σ can be defined existentially from timelike relatedness τ by $\exists^3$-formula $\hat{\mathcal{E}}_{\tau\to\sigma}$ in (Q^n, τ).*
2. *Lightlike relatedness λ can be defined universally from timelike relatedness τ by $\forall^3$-formula $\hat{\mathcal{U}}_{\tau\to\lambda}$ in (Q^n, τ).*

Proof. It is enough to prove Item 1 because Item 2 follows from Item 1 and the fact that exactly one of relations $\bar{\mathsf{p}}\ \tau\ \bar{\mathsf{q}}$, $\bar{\mathsf{p}}\ \lambda\ \bar{\mathsf{q}}$, $\bar{\mathsf{p}}\ \sigma\ \bar{\mathsf{q}}$, and $\bar{\mathsf{p}} = \bar{\mathsf{q}}$ holds for every $\bar{\mathsf{p}}, \bar{\mathsf{q}} \in Q^n$.

To prove Item 1, let us assume first that $\hat{\mathcal{E}}_{\tau\to\sigma}(\bar{\mathsf{p}}, \bar{\mathsf{q}})$ holds for some $\bar{\mathsf{p}}, \bar{\mathsf{q}} \in Q^n$ and show that then $\bar{\mathsf{p}}$ and $\bar{\mathsf{q}}$ has to be spacelike related. Let $\bar{\mathsf{x}}$, $\bar{\mathsf{y}}$, and $\bar{\mathsf{z}}$ be such points that they show the validity of $\hat{\mathcal{E}}_{\tau\to\sigma}(\bar{\mathsf{p}}, \bar{\mathsf{q}})$. Clearly, $\bar{\mathsf{p}}$ cannot be equal to $\bar{\mathsf{q}}$ since $\bar{\mathsf{z}}\ \bar{\tau}_{\neq}\ \bar{\mathsf{p}}$ but $\bar{\mathsf{z}}\ \tau\ \bar{\mathsf{q}}$.

Let us now show that points $\bar{\mathsf{p}}$ and $\bar{\mathsf{q}}$ cannot be timelike or lightlike related. If they were such, we could assume, without loosing generality, that $\bar{\mathsf{q}}$ is in the causal past of $\bar{\mathsf{p}}$. Then $\bar{\mathsf{z}}$ should be in the timelike future of $\bar{\mathsf{q}}$ otherwise $\bar{\mathsf{z}}$ and $\bar{\mathsf{p}}$ were timelike related contrary to the requirement $z\ \bar{\tau}_{\neq}\ p$ of $\hat{\mathcal{E}}_{\tau\to\sigma}$. For analogous reasons, the relation being in the future/past should alternate between the consecutive points along the circle $\bar{\mathsf{q}} - \bar{\mathsf{z}} - \bar{\mathsf{y}} - \bar{\mathsf{x}} - \bar{\mathsf{p}} - \bar{\mathsf{q}}$ which is not possible because there are odd many edges between them, i.e., $\bar{\mathsf{y}}$ has to be in the past of $\bar{\mathsf{z}}$, $\bar{\mathsf{x}}$ has to be in the future of $\bar{\mathsf{y}}$, $\bar{\mathsf{p}}$ has to be in the past of $\bar{\mathsf{x}}$, and finally $\bar{\mathsf{q}}$ has to be in the future of $\bar{\mathsf{p}}$ contradicting that $\bar{\mathsf{q}}$ is in the past of $\bar{\mathsf{p}}$. Hence $\bar{\mathsf{p}}$ and $\bar{\mathsf{q}}$ has to be spacelike related.

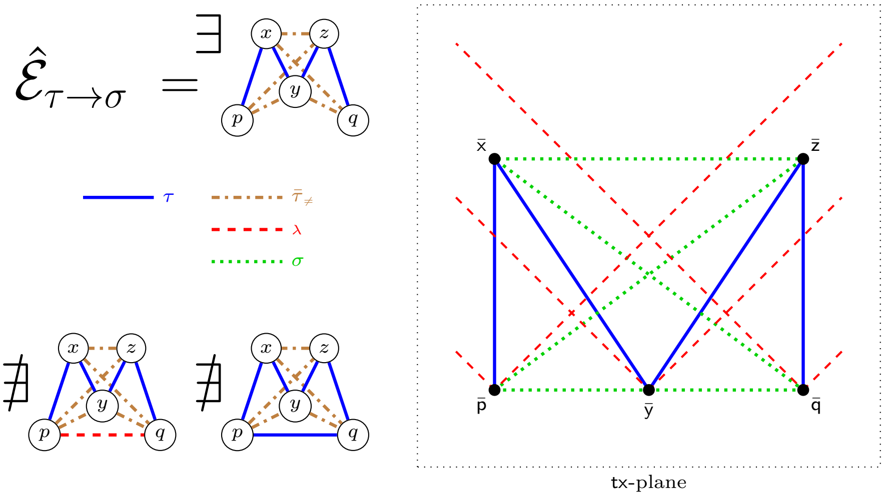

Fig. 15. The figure illustrates the proof of that the relation defined by $\exists^3$-formula $\hat{\mathcal{E}}_{\tau\to\sigma}$ defines σ if $n = 2$ or $(Q, +, \cdot, \leq)$ is a Euclidean field.

To show the other direction, let $\bar{\mathsf{p}}$ and $\bar{\mathsf{q}}$ be two arbitrary spacelike related points. We can assume that $\bar{\mathsf{p}}$ and $\bar{\mathsf{q}}$ are the horizontally related, in the tx-plane by the automorphism argument, cf. Proposition 1. Then we can choose $\bar{\mathsf{y}}$ to be the midpoint of $\bar{\mathsf{p}}$ and $\bar{\mathsf{q}}$, and choose $\bar{\mathsf{x}}$ and $\bar{\mathsf{z}}$ respectively above $\bar{\mathsf{p}}$ and $\bar{\mathsf{q}}$ high enough but not too high, cf. Fig. 15. □

Remark 6 *($n \geq 3$).* Regarding Theorem 8, the proof of the direction showing that σ contains the relation defined by $\hat{\mathcal{E}}_{\tau\to\sigma}$ works over arbitrary ordered field. The other direction can be generalized to Archimedean ordered fields using the facts that (a) every Archimedean ordered field is isomorphic to a subfield of the reals and (b) every subfield of the reals is dense in the field of reals. However, this proof idea does not work in general, and hence, we do not know whether it is possible that $\hat{\mathcal{E}}_{\tau\to\sigma}$ defines a nonempty relation strictly smaller than σ in some non-Archimedean ordered fields.

Let us now replace τ with σ in formulas $\hat{\mathcal{E}}_{\tau\to\sigma}$ and $\hat{\mathcal{U}}_{\tau\to\lambda}$ and see when do they work as respective definitions of τ and λ. So let

$$\hat{\mathcal{E}}_{\sigma\to\tau}(p,q) \stackrel{\text{def}}{=} \exists x \exists y \exists z (x\ \sigma\sigma\bar{\sigma}_{\neq}\bar{\sigma}_{\neq}\ pyqz\ ,\ y\ \bar{\sigma}_{\neq}\bar{\sigma}_{\neq}\sigma\ pqz\ ,\ z\ \bar{\sigma}_{\neq}\sigma\ pq)$$

and let

$$\hat{\mathcal{U}}_{\sigma\to\lambda}(p,q) \stackrel{\text{def}}{=} \neg\hat{\mathcal{E}}_{\sigma\to\tau}(p,q)\ ,\ p \neq q\ ,\ p\ \bar{\sigma}\ q.$$

As before, by the isomorphism between (Q^2, τ, σ) and (Q^2, σ, τ), we get the following corollary of Theorem 8.

Corollary 4 *($n = 2$). Let $(Q, +, \cdot, \leq)$ be an arbitrary ordered field, then:*

1. *Timelike relatedness τ can be defined existentially from spacelike relatedness σ by $\exists^3$-formula $\hat{\mathcal{E}}_{\sigma\to\tau}$ in (Q^2, σ).*
2. *Lightlike relatedness λ can be defined universally from spacelike relatedness σ by $\forall^3$-formula $\hat{\mathcal{U}}_{\sigma\to\lambda}$ in (Q^2, σ).* □

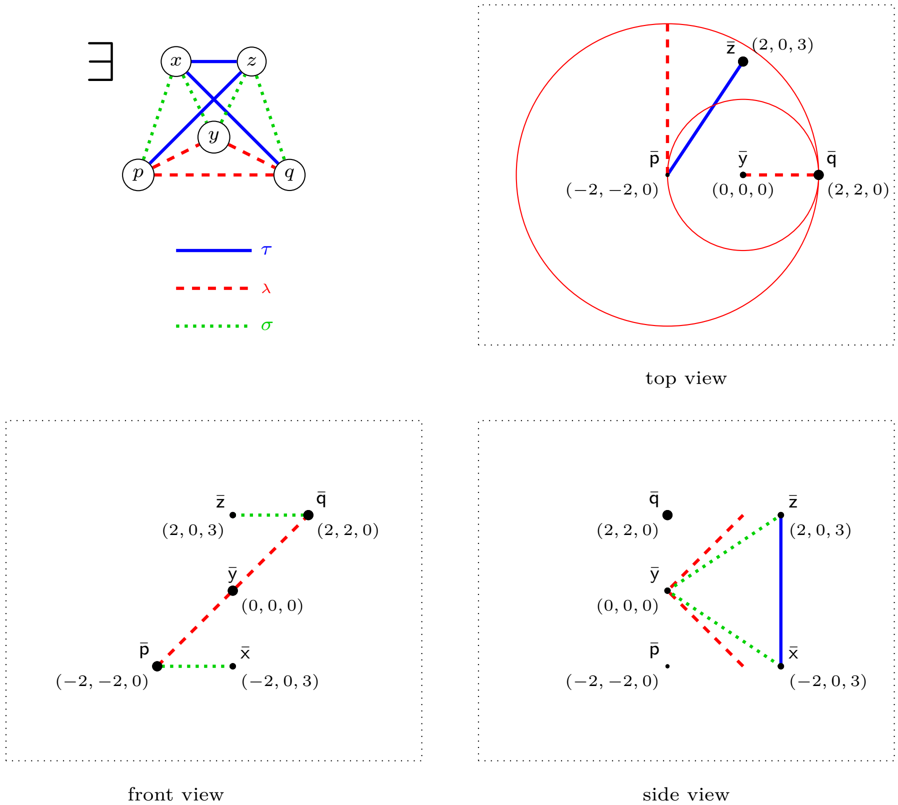

Fig. 16. This figure illustrates why $\exists^3$-formula $\hat{\mathcal{E}}_{\sigma\to\tau}$ does not define timelike relatedness τ in (Q^n,σ) if $n \geq 3$.

Remark 7 *($n \geq 3$).* The assumption $n = 2$ cannot be omitted from Corollary 4, i.e., $\exists^3$-formula $\hat{\mathcal{E}}_{\sigma\to\tau}$ does not define τ in (Q^n,σ) if $n \geq 3$. It is enough to see this in case $n = 3$ since (Q^3,σ) can be embedded as a submodel to (Q^n,σ) for all $n \geq 3$. To show it for $n = 3$, consider the following lightlike related points $\bar{\mathsf{p}} = (-2,-2,0)$ and $\bar{\mathsf{q}} = (2,2,0)$. It is straightforward to check that points $\bar{\mathsf{x}} = (-2,0,3)$, $\bar{\mathsf{y}} = (0,0,0)$ and $\bar{\mathsf{z}} = (2,0,3)$ are such that they show the validity of $\hat{\mathcal{E}}_{\sigma\to\tau}(\bar{\mathsf{p}},\bar{\mathsf{q}})$, see Fig. 16 for a picturesque justification. Hence, $\forall^4$-formula $\hat{\mathcal{U}}_{\sigma\to\lambda}$ does not define λ if $n \geq 3$ because the relation defined by it does not contain lightlike related point pair $\bar{\mathsf{p}} = (-2,-2,0)$ and $\bar{\mathsf{q}} = (2,2,0)$.

Theorem 9 *($n \geq 2$). Let $n \geq 2$. Relation λ is not definable existentially from τ in (Q^n,τ). Similarly, λ is not definable existentially from σ in (Q^n,σ). Thus, σ is not definable universally from τ in (Q^n,τ) and τ is not definable universally from σ in (Q^n,σ).*

Proof. Let formula $\varphi(x,y)$ be an arbitrary existential formula in the language of (Q^n,τ). Then $\varphi(x,y)$ is logically equivalent to a formula of the following form

$$\exists z_1 \exists z_2 \dots \exists z_k \; B(x,y,z_1,\dots,z_k)$$

such that B is a Boolean combination of relations τ and $=$ between variables $z_1,\dots,z_k$, x and y.

We are going to show that if the relation defined by φ holds for some lightlike related points $\bar{\mathsf{x}}$ and $\bar{\mathsf{y}}$, then it also holds for some spacelike related points $\bar{\mathsf{x}}'$ and $\bar{\mathsf{y}}'$, and hence φ cannot be a definition of lightlike relatedness. To see this, let $\bar{\mathsf{x}}$ and $\bar{\mathsf{y}}$ be lightlike related points of Q^n for which $\varphi(\bar{\mathsf{x}},\bar{\mathsf{y}})$ holds. Then there are points $\bar{\mathsf{z}}_1,\dots,\bar{\mathsf{z}}_k \in Q^n$ such that $B(\bar{\mathsf{x}},\bar{\mathsf{y}},\bar{\mathsf{z}}_1,\dots,\bar{\mathsf{z}}_k)$ holds. There is a small enough $\varepsilon > 0$ such that map

$$T_\varepsilon : (\mathsf{r}_0,\mathsf{r}_1,\dots,\mathsf{r}_{n-1}) \mapsto ((1-\varepsilon)\mathsf{r}_0,\mathsf{r}_1,\dots,\mathsf{r}_{n-1})$$

scaling time down does not change the τ, $\bar{\tau}$, $=$ and $\neq$ relations between points $\bar{\mathsf{z}}_1, \ldots, \bar{\mathsf{z}}_k$, $\bar{\mathsf{x}}$ and $\bar{\mathsf{y}}$. That this is so can be seen as follows. For fixed timelike related points $\bar{\mathsf{p}}$ and $\bar{\mathsf{q}}$, there is a $\delta > 0$ such that $T_\varepsilon(\bar{\mathsf{p}})$ and $T_\varepsilon(\bar{\mathsf{q}})$ are also timelike related for all $0 < \varepsilon < \delta$. Since there are only finitely many pairs of points to consider, there is an appropriate ε preserving the timelike relatedness relations between points $\bar{\mathsf{x}}$, $\bar{\mathsf{y}}$, $\bar{\mathsf{z}}_1, \ldots,$ $\bar{\mathsf{z}}_k$. Let us fix such an ε. Then map T_ε takes $\bar{\tau}$-related points to $\bar{\tau}$-related ones because it decreases the time difference but does not change the spatial distance. Also T_ε maps different points to different ones because it is a bijection. Let $\bar{\mathsf{z}}'_1, \ldots, \bar{\mathsf{z}}'_k$, $\bar{\mathsf{x}}'$ and $\bar{\mathsf{y}}'$ be the T_ε-images of points $\bar{\mathsf{z}}_1, \ldots, \bar{\mathsf{z}}_k$, $\bar{\mathsf{x}}$, and $\bar{\mathsf{y}}$, respectively. Then $B(\bar{\mathsf{x}}', \bar{\mathsf{y}}', \bar{\mathsf{z}}'_1, \ldots, \bar{\mathsf{z}}'_k)$ holds by the above properties of T_ε, and hence $\varphi(\bar{\mathsf{x}}', \bar{\mathsf{y}}')$ also holds. We also have that $\bar{\mathsf{x}}'$ and $\bar{\mathsf{y}}'$ are spacelike related because T_ε decreases the time difference but does not change the spatial distance. This completes the proof of that relation λ is not definable existentially from τ in (Q^n, τ).

The proof of that λ is not definable existentially from σ in (Q^n, σ) is completely analogous but using a map scaling time up.

Finally, that σ is not definable universally from τ in (Q^n, τ) and τ is not definable universally from σ in (Q^n, σ) follows as before from that exactly one of relations τ, λ, σ and $=$ holds between any two points. □

Theorem 10 *($n \geq 2$). Let $n \geq 2$. Neither timelike relatedness τ nor spacelike relatedness σ is definable existentially or universally from lightlike relatedness λ in (Q^n, λ).*

We have already seen that Theorem 10 holds in case $n = 2$, see Remark 2. Now we are going to give a proof that works for all $n \geq 2$.

Proof. As in the proof of Theorem 6 and for the very same reason, is enough to show the nonexistence of existential definitions. To show the nonexistence of existential definitions, let us consider the following map:

$$h : (\mathsf{r}_0, \mathsf{r}_1, \ldots, \mathsf{r}_{n-1}) \mapsto \frac{1}{\mathsf{r}_0^2 - \mathsf{r}_1^2 - \ldots - \mathsf{r}_{n-1}^2}(\mathsf{r}_0, \mathsf{r}_1, \ldots, \mathsf{r}_{n-1}).$$

It is known, see e.g., [9], and also straightforward to check that h is a bijection of the complement of the light cone through the origin $\bar{\mathsf{o}} = (0, 0, \ldots, 0)$ to itself and h maps lightlike related pairs of points to lightlike related ones. By the definition of h, we also have that $\bar{\mathsf{r}}$, $\bar{\mathsf{o}}$ and $h(\bar{\mathsf{r}})$ are on the same line; h leaves the hyperboloid defined by equation $\mathsf{r}_0^2 - \mathsf{r}_1^2 - \ldots - \mathsf{r}_{n-1}^2 = 1$ pointwise fixed; h maps every point of the hyperboloid defined by equation $\mathsf{r}_0^2 - \mathsf{r}_1^2 - \ldots - \mathsf{r}_{n-1}^2 = -1$ to its opposite; and h maps the region inside these hyperboloids to the region outside of them.

Let $\varphi(p, q)$ be an arbitrary existential formula in the language of structure (Q^n, λ). Then $\varphi(p, q)$ is of the form $\exists z_1 \exists z_2 \ldots \exists z_k\, B(p, q, z_1, \ldots, z_k)$, where B is a Boolean combination of relations λ and $=$ between variables $z_1, \ldots, z_k$, p and q.

Using the fact that h is a bijection preserving λ, we are going to show that the relation defined by φ holds for some timelike related points iff it holds for some spacelike related points. So φ can define neither τ nor σ.

Let $\bar{\mathsf{p}}$ and $\bar{\mathsf{q}}$ be points such that $\varphi(\bar{\mathsf{p}}, \bar{\mathsf{q}})$ holds. Then there are points $\bar{\mathsf{z}}_1, \ldots, \bar{\mathsf{z}}_k$ such that $B(\bar{\mathsf{p}}, \bar{\mathsf{q}}, \bar{\mathsf{z}}_1, \ldots, \bar{\mathsf{z}}_k)$ holds. Using automorphisms of (Q^n, λ), we are going to show that we can assume, without loosing generality, that

1. points $\bar{\mathsf{z}}_1, \ldots, \bar{\mathsf{z}}_k$, $\bar{\mathsf{p}}$, and $\bar{\mathsf{q}}$ are all in the domain of h,
2. if $\bar{\mathsf{p}}\ \tau\ \bar{\mathsf{q}}$, then $\bar{\mathsf{p}} = (0, 1, 0, \ldots, 0)$ and there is a $\mathsf{t} > 0$ such that $\bar{\mathsf{q}} = (\mathsf{t}, 1, 0, \ldots, 0)$ and $\mathsf{t}^2 - 1 > 1$ (i.e., $\bar{\mathsf{q}}$ is above the upper half of the hyperbola $\mathsf{t}^2 - \mathsf{x}^2 = 1$ in the tx-plane),
3. if $\bar{\mathsf{p}}\ \sigma\ \bar{\mathsf{q}}$, then $\bar{\mathsf{p}} = (1, 0, \ldots, 0)$ and there is an $\mathsf{x} > 0$ such that $\bar{\mathsf{q}} = (1, \mathsf{x}, 0, \ldots, 0)$ and $1 - \mathsf{x}^2 < -1$ (i.e., $\bar{\mathsf{q}}$ is on the right of the right half of the hyperbola $\mathsf{t}^2 - \mathsf{x}^2 = -1$ in the tx-plane).

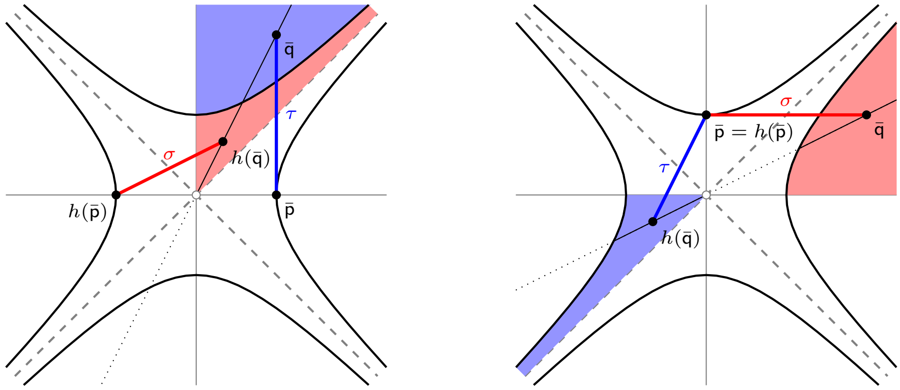

Fig. 17. The figure illustrates that h interchanges the relations τ and σ between points $\bar{\mathsf{p}}$ and $\bar{\mathsf{q}}$ if they are from the appropriate regions given in the proof of Theorem 10.

In this setting, the h-images of $\bar{\mathsf{p}}$ and $\bar{\mathsf{q}}$ are timelike if $\bar{\mathsf{p}}$ and $\bar{\mathsf{q}}$ are spacelike related, and they are spacelike related if $\bar{\mathsf{p}}$ and $\bar{\mathsf{q}}$ are timelike related, see Fig. 17. Since h preserves the λ, $\bar{\lambda}$, $=$ and $\neq$ relations between points $\bar{\mathsf{z}}_1, \ldots, \bar{\mathsf{z}}_k$, $\bar{\mathsf{p}}$, and $\bar{\mathsf{q}}$, this shows that there are timelike related points in φ relation exactly if there are spacelike related ones in φ relation.

The only remaining thing to show is that points $\bar{\mathsf{z}}_1, \ldots, \bar{\mathsf{z}}_k$, $\bar{\mathsf{p}}$, and $\bar{\mathsf{q}}$ can indeed be transformed by an automorphism of (Q^n, λ) satisfying 1, 2 and 3. By Proposition 1, we can transform them such that $\bar{\mathsf{p}}$ and $\bar{\mathsf{q}}$ are situated appropriately in the tx-plane. The only problem is that maybe some of points $\bar{\mathsf{z}}_1, \ldots, \bar{\mathsf{z}}_k$ are not in the domain of h. If that is so, then by applying an appropriate automorphism α enough many times, we can move everything into the domain of h while keeping points $\bar{\mathsf{p}}$ and $\bar{\mathsf{q}}$ in the right places.

In case $\bar{\mathsf{p}}\ \tau\ \bar{\mathsf{q}}$, we can choose this automorphism α to be the composition of a uniform scaling (say by factor 2) and an appropriate horizontal translation (in the negative direction along the x-axis). Of course one such step may create new problematic points, but iterating this step will put every points to the right place after finitely many iterations because there are only finitely many points to deal with and a point cannot become problematic more than twice. The latter is so because the sequence of points $\bar{\mathsf{z}}$, $\alpha(\bar{\mathsf{z}})$, $\alpha\big(\alpha(\bar{\mathsf{z}})\big)$, … are different points on the same line, and a line cannot intersect the set of problematic points (i.e., the light cone through the origin) more than twice.[10]

In case $\bar{\mathsf{p}}\ \sigma\ \bar{\mathsf{q}}$, the same idea works but here after scaling up we should translate the points downwards (i.e., in the negative direction along the t-axis). □

Based on the definitions given by Winnie [18], we get the following formulas which respectively define λ and τ from σ:

$$\mathcal{W}_{\sigma\to\lambda}(x,y) \overset{\mathsf{def}}{=} x\ \bar{\sigma}\ y\,,\ x \neq y\,,\ \forall u \forall v \exists z_u \exists z_v (z_u\ \bar{\sigma}\sigma\sigma\ uxy \vee z_v\ \bar{\sigma}\sigma\sigma\ vxy \vee u\ \bar{\sigma}\ v)$$

$$\mathcal{W}_{\sigma\to\tau}(x,y) \overset{\mathsf{def}}{=} \neg\mathcal{W}_{\sigma\to\lambda}(x,y)\,,\ x\ \bar{\sigma}\ y\,,\ x \neq y.$$

Theorem 11 *($n \geq 2$; (Eucl.) or $n = 2$). Assume that $n = 2$ or that $(Q, +, \cdot, \leq)$ is a Euclidean field. Then in model (Q^n, σ), lightlike relatedness λ can be defined from spacelike relatedness σ by $\forall^2\exists^2$-formula $\mathcal{W}_{\sigma\to\lambda}$. Hence timelike relatedness τ can be also defined from σ by $\exists^2\forall^2$-formula $\mathcal{W}_{\sigma\to\tau}$.*

Proof. Let us first show that if points $\bar{\mathsf{x}}$ and $\bar{\mathsf{y}}$ are lightlike related, then they satisfy formula $\mathcal{W}_{\sigma\to\lambda}$. So let $\bar{\mathsf{x}}$ and $\bar{\mathsf{y}}$ be arbitrary two lightlike related points. Since then $\bar{\mathsf{x}}$ and $\bar{\mathsf{y}}$ are clearly neither equal nor spacelike

[10] More precisely, the considered automorphism α maps point $\bar{\mathsf{z}} = (z_0, z_1, z_2, \ldots z_{n-1})$ to $(2z_0, 2z_1 - 1, 2z_2, \ldots, 2z_{n-1})$. Hence, points $\bar{\mathsf{z}}$, $\alpha(\bar{\mathsf{z}})$ and $\alpha\big(\alpha(\bar{\mathsf{z}})\big)$ are on the same line because $\alpha\big(\alpha(\bar{\mathsf{z}})\big) - \alpha(\bar{\mathsf{z}}) = 2\big(\alpha(\bar{\mathsf{z}}) - \bar{\mathsf{z}}\big)$, which can be checked by a straightforward calculation. Checking that this α keeps points $\bar{\mathsf{p}}$ and $\bar{\mathsf{q}}$ in the right places is also straightforward.

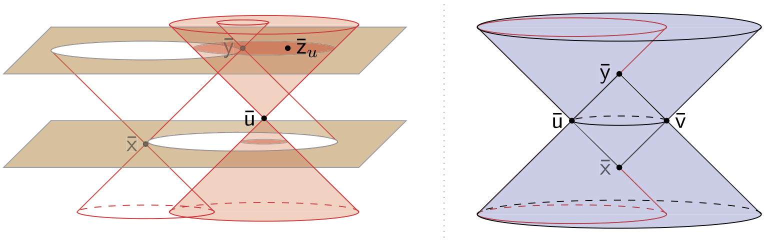

Fig. 18. Illustration for the proof of Theorem 11.

related, we only have to show that, for all points $\bar{u}$ and $\bar{v}$, there are points $\bar{z}_u$ and $\bar{z}_v$ such that $\bar{z}_u\ \bar{\sigma}\ \bar{u}$, $\bar{z}_u\ \sigma\ \bar{x}$ and $\bar{z}_u\ \sigma\ \bar{y}$ hold, $\bar{z}_v\ \bar{\sigma}\ \bar{v}$, $\bar{z}_v\ \sigma\ \bar{x}$ and $\bar{z}_v\ \sigma\ \bar{y}$ hold or $\bar{u}$ and $\bar{v}$ are not spacelike related. Unless $\bar{u}$ is in the segment $\bar{x}\bar{y}$, there is an appropriate $\bar{z}_u$. This can be checked by considering the horizontal hyperplanes through $\bar{x}$ and $\bar{y}$, see the left hand side of Fig. 18. The light cone through $\bar{u}$ intersects these hyperplanes in two spheres (at most one of which is degenerate as $\bar{u}$ cannot be in both hyperplanes). It is straightforward to check that there is a point $\bar{z}_u$ inside one of these spheres which is spacelike related to both $\bar{x}$ and $\bar{y}$, unless $\bar{u}$ is in the lightlike segment $\bar{x}\bar{y}$. Similarly, unless $\bar{v}$ is in the segment $\bar{x}\bar{y}$, there is an appropriate $\bar{z}_v$. If both $\bar{u}$ and $\bar{v}$ are in the lightlike segment $\bar{x}\bar{y}$, then they are either equal or lightlike related, and hence $\bar{u}\ \bar{\sigma}\ \bar{v}$ holds. So lightlike relatedness implies $\mathcal{W}_{\sigma\to\lambda}$-relatedness.

Let us now show that if points $\bar{x}$ and $\bar{y}$ are $\mathcal{W}_{\sigma\to\lambda}$-related, then they have to be lightlike related. Clearly, $\bar{x}$ and $\bar{y}$ have to be different and cannot be spacelike related. Thus, we only have to show that they cannot be timelike related. So let $\bar{x}$ and $\bar{y}$ be two arbitrary timelike related points. We should show that they are not $\mathcal{W}_{\sigma\to\lambda}$-related.

By Proposition 1, we can assume that $\bar{x}$ and $\bar{y}$ are vertically related and $\bar{y}$ is in the (causal) future of $\bar{x}$. To show that they are not $\mathcal{W}_{\sigma\to\lambda}$-related, we should find spacelike related points $\bar{u}$ and $\bar{v}$ such that at least one of $\bar{z}_u\ \sigma\ \bar{u}$, $\bar{z}_u\ \bar{\sigma}\ \bar{x}$ and $\bar{z}_u\ \bar{\sigma}\ \bar{y}$ holds and at least one of $\bar{z}_v\ \sigma\ \bar{v}$, $\bar{z}_v\ \bar{\sigma}\ \bar{x}$ and $\bar{z}_v\ \bar{\sigma}\ \bar{y}$ holds for all points $\bar{z}_u$ and $\bar{z}_v$. Let $\bar{u}$ and $\bar{v}$ be spacelike related points such that they are both in the lightlike past of $\bar{y}$ and in the lightlike future of $\bar{x}$. There are such points, see the right hand side of Fig. 18. Then the causal future of $\bar{u}$ is in the causal future of $\bar{x}$ and the causal past of $\bar{u}$ is in the causal past of $\bar{y}$ by the transitivity of the causal past and future relations. Hence, if $\bar{z}\ \bar{\sigma}\ \bar{u}$, then $\bar{z}\ \bar{\sigma}\ \bar{x}$ or $\bar{z}\ \bar{\sigma}\ \bar{y}$. So every point is $\bar{\sigma}$-related to $\bar{x}$ or $\bar{y}$, or it is σ-related to $\bar{u}$; and the same holds for $\bar{v}$. Thus $\bar{x}$ and $\bar{y}$ are not $\mathcal{W}_{\sigma\to\lambda}$-related, and this is what we wanted to prove.

From this, it follows that $\mathcal{W}_{\sigma\to\tau}$ defines timelike relatedness as in the previous proofs because exactly one of relations τ, λ, σ and $=$ holds. □

Let $\mathcal{W}^{[\tau/\sigma]}_{\sigma\to\lambda}$ and $\mathcal{W}^{[\tau/\sigma]}_{\sigma\to\tau}$ be the formulas that we respectively get from $\mathcal{W}_{\sigma\to\lambda}$ and $\mathcal{W}_{\sigma\to\tau}$ when replacing σ with τ. By the isomorphism of (Q^2, τ, λ) and (Q^2, σ, λ), the following is a corollary of Theorem 11:

Corollary 5 *($n = 2$). Let $(Q, +, \cdot, \leq)$ be an arbitrary ordered field. Then in model (Q^2, τ), lightlike relatedness λ can be defined from timelike relatedness τ by $\forall^2\exists^2$-formula $\mathcal{W}^{[\tau/\sigma]}_{\sigma\to\lambda}$. Hence spacelike relatedness σ can be also defined from τ by $\exists^2\forall^2$-formula $\mathcal{W}^{[\tau/\sigma]}_{\sigma\to\tau}$.*

Remark 8. Even if the field $(Q, +, \cdot, \leq)$ is Euclidean, the statement of Corollary 5 does not hold if the dimension $n > 2$. In this case, formulas $\mathcal{W}^{[\tau/\sigma]}_{\sigma\to\lambda}$ and $\mathcal{W}^{[\tau/\sigma]}_{\sigma\to\tau}$ respectively define relations $\bar{\tau}_{\neq}$ and $\emptyset$. This is so because formula

$$\phi(x, y) := \forall u \forall v \exists z_u \exists z_v (z_u\ \bar{\tau}\tau\tau\ uxy \vee z_v\ \bar{\tau}\tau\tau\ vxy \vee u\ \bar{\tau}\ v)$$

defines $\bar{\tau}$, which is the union of relations σ, λ and $=$. That the relation defined by ϕ contains σ can be shown based on the following observation: If point $\bar{u}$ is not simultaneous to horizontally related points $\bar{x}$

Table 2
The table summarizes the results and open problems when the dimension $n = 2$.

$n = 2$	$\tau \to \sigma$	$\tau \to \lambda$	$\sigma \to \tau$	$\sigma \to \lambda$	$\lambda \to \tau$	$\lambda \to \sigma$
$\exists^2$ / $\forall^2$	$\nexists$ (not possible)					
$\exists^3$	$\hat{\mathcal{E}}_{\tau\to\sigma}$	$\nexists$	$\hat{\mathcal{E}}_{\sigma\to\tau}$	$\nexists$ (not possible)		
$\forall^3$	$\nexists$	$\hat{\mathcal{U}}_{\tau\to\lambda}$	$\nexists$	$\hat{\mathcal{U}}_{\sigma\to\lambda}$	$\nexists$	
$\exists^4$	$\mathcal{E}_{\tau\to\sigma}$	$\nexists$	$\mathcal{E}_{\sigma\to\tau}$	$\nexists$ (not possible)		
$\forall^4$	$\nexists$	$\mathcal{U}_{\tau\to\lambda}$	$\nexists$	$\mathcal{U}_{\sigma\to\lambda}$	$\nexists$	
$\exists^1\forall^1$	?	$\Psi_{\tau\to\lambda}$	?	$\Psi_{\sigma\to\lambda}$	$\nexists$	
$\forall^1\exists^1$	$\Psi_{\tau\to\sigma}$	?	$\Psi_{\sigma\to\tau}$	?	$\nexists$	
$\exists^2\forall^1$ / $\exists^1\forall^2$	?	✓	?	✓	$\nexists$	
$\forall^2\exists^1$ / $\forall^1\exists^2$	✓	?	✓	?	$\nexists$	
$\exists^2\forall^2$	$\mathcal{W}^{[\tau/\sigma]}_{\sigma\to\tau}$	✓	$\mathcal{W}_{\sigma\to\tau}$	✓	$\nexists$	
$\forall^2\exists^2$	✓	$\mathcal{W}^{[\tau/\sigma]}_{\sigma\to\lambda}$	✓	$\mathcal{W}_{\sigma\to\lambda}$	$\nexists$	

Table 3
The table summarizes the results and open problems if $n > 2$ and the underlying field is Euclidean.

$n > 2$, (Eucl.)	$\tau \to \sigma$	$\tau \to \lambda$	$\sigma \to \tau$	$\sigma \to \lambda$	$\lambda \to \tau$	$\lambda \to \sigma$
$\exists^2$ or $\forall^2$	$\nexists$ (not possible)					
$\exists^3$	$\hat{\mathcal{E}}_{\tau\to\sigma}$	$\nexists$	?	$\nexists$ (not possible)		
$\forall^3$	$\nexists$	$\hat{\mathcal{U}}_{\tau\to\lambda}$	$\nexists$	?	$\nexists$	
$\exists^4$	$\mathcal{E}_{\tau\to\sigma}$	$\nexists$	?	$\nexists$ (not possible)		
$\forall^4$	$\nexists$	$\mathcal{U}_{\tau\to\lambda}$	$\nexists$	?	$\nexists$	
$\exists^*$	✓	$\nexists$	?	$\nexists$ (not possible)		
$\forall^*$	$\nexists$	✓	$\nexists$	?	$\nexists$	
$\exists^1\forall^1$	?	$\Psi_{\tau\to\lambda}$	?	$\Psi_{\sigma\to\lambda}$	?	$\Psi_{\lambda\to\sigma}$
$\forall^1\exists^1$	$\Psi_{\tau\to\sigma}$	?	$\Psi_{\sigma\to\tau}$	?	$\Psi_{\lambda\to\tau}$	?
$\exists^2\forall^1$ / $\exists^1\forall^2$	?	✓	?	✓	?	✓
$\forall^2\exists^1$ / $\forall^1\exists^2$	✓	?	✓	?	✓	?
$\exists^2\forall^2$	?	✓	$\mathcal{W}_{\sigma\to\tau}$	✓	?	✓
$\forall^2\exists^2$	✓	?	✓	$\mathcal{W}_{\sigma\to\lambda}$	✓	?
$\exists^*\forall^*$	✓	✓	✓	✓	?	✓
$\forall^*\exists^*$	✓	✓	✓	✓	✓	?

and $\bar{\mathsf{y}}$, then after (or before) a certain time the horizontal slices of the light cone through $\bar{\mathsf{u}}$ do not cover the corresponding horizontal slices of the intersection of the light cones through $\bar{\mathsf{x}}$ and $\bar{\mathsf{y}}$, and hence there is an appropriate point $\bar{\mathsf{z}}_u$ for which $\bar{\mathsf{z}}_u\ \bar{\tau}\tau\tau\ \bar{\mathsf{u}}\bar{\mathsf{x}}\bar{\mathsf{y}}$ holds.

5. Concluding remarks and open problems

We have carefully investigated the interdefinability between timelike, lightlike and spacelike relatedness of Minkowski spacetime in various dimensions over Euclidean fields and in some cases over arbitrary ordered fields. We aimed to find the simplest possible definitions. We have shown that in terms of number of auxiliary variables 2 is minimal, but in terms of quantifier complexity, we left some natural questions open.

For all $n \geq 2$, it is open if universal-existential definitions $\mathcal{W}^{[\tau/\sigma]}_{\sigma\to\tau}$ and $\mathcal{W}_{\sigma\to\tau}$ (and existential-universal definitions $\mathcal{W}^{[\tau/\sigma]}_{\sigma\to\lambda}$ and $\mathcal{W}_{\sigma\to\lambda}$) are minimal in the number of used auxiliary variables; in other words, it is open if there are corresponding universal-existential (existential-universal) definitions using fewer quantifiers. In the case $n > 2$, it is open if there is an existential-universal definition of timelike relatedness from lightlike relatedness which works (at least) over Euclidean ordered field, or equivalently if there is a universal-existential definition of spacelike relatedness from lightlike relatedness.

In Tables 2 and 3, we summarize our results and open problems related to the simplest possible quantifier complexity. For example, we do not know whether $\mathcal{W}_{\sigma\to\tau}$ and $\mathcal{W}^{[\tau/\sigma]}_{\sigma\to\tau}$ are the simplest existential-universal definitions or not, see Table 2. In case the dimension $n \geq 3$, we also do not know if timelike relatedness can or cannot be defined from lightlike relatedness by an existential-universal formula, or equivalently if spacelike relatedness can be defined from lightlike relatedness by a universal-existential formula, see Table 3.

Here we were using a quite direct approach both to find defining formulas and to show their nonexistence. A direct way of learning the existence or nonexistence of $\exists^n\forall^k$ and $\forall^n\exists^k$ definitions in the missing cases for each fixed n and k would be to check all the finitely many candidates that one can have in conjunctive prenex normal form. For learning the answers in the cases where the numbers of the existential and universal quantifiers are not fixed, one could try to use preservation theorems, e.g., one could try to use Łoś–Tarski theorem (see, e.g., [6, Thm. 6.5.4], cf. also [16]).

References

[1] H. Andréka, J.X. Madarász, I. Németi, G. Székely, Description of the algebra of definable relations of special relativity theory, in preparation.
[2] B. Bennett, I. Düntsch, Axioms, algebras and topology, in: Handbook of Spatial Logics, Springer, 2007, pp. 99–159.
[3] L. Cocco, J. Babic, A system of axioms for Minkowski spacetime, J. Philos. Log. 50 (1) (Feb. 2021) 149–185.
[4] R. Goldblatt, Orthogonality and Spacetime Geometry, Springer–Verlag, Berlin, 1987.
[5] S.W. Hawking, A.R. King, P.J. McCarthy, A new topology for curved space–time which incorporates the causal, differential, and conformal structures, J. Math. Phys. 17 (2) (1976) 174–181.
[6] W. Hodges, Model Theory, Cambridge University Press, 1993.
[7] E.H. Kronheimer, R. Penrose, On the structure of causal spaces, Math. Proc. Camb. Philos. Soc. 63 (2) (1967) 481–501.
[8] R.W. Latzer, Nondirected light signals and the structure of time, Synthese 24 (1) (Mar. 1972) 236–280.
[9] J.A. Lester, Chapter 16 - distance preserving transformations, in: F. Buekenhout (Ed.), Handbook of Incidence Geometry, North-Holland, Amsterdam, 1995, pp. 921–944.
[10] D.B. Malament, Causal theories of time and the conventionality of simultaneity, Noûs 11 (3) (1977) 293–300.
[11] D.B. Malament, The class of continuous timelike curves determines the topology of spacetime, J. Math. Phys. 18 (7) (1977) 1399–1404.
[12] B. Mundy, Optical axiomatization of Minkowski space-time geometry, Philos. Sci. 53 (1) (1986) 1–30.
[13] V. Pambuccian, Alexandrov-Zeeman type theorems expressed in terms of definability, Aequ. Math. 74 (3) (2007) 249–261.
[14] A.A. Robb, A theory of time and space, in: Geometry of Time and Space, revised edition, Cambridge University Press, 1914, published in 1936.
[15] A.A. Robb, Geometry of Time and Space, Cambridge University Press, 1936.
[16] A. Sankaran, B. Adsul, S. Chakraborty, A generalization of the Łoś–Tarski preservation theorem, Ann. Pure Appl. Log. 167 (3) (2016) 189–210.
[17] A. Tarski, S. Givant, A Formalization of Set Theory Without Variables, Colloquium Publications, vol. 41, American Mathematical Society, 1987.
[18] J.A. Winnie, The causal theory of space-time, in: J.S. Earman, C.N. Glymour, J.J. Stachel (Eds.), Foundation of Space-Time Theories, 1977, pp. 134–205.