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The Classical Representations of the Conceptual Hierarchies with Graphs

Conceptual hierarchies are often presented in graphs. These typical representations or containment or derivation relationships are built. The classic visual depictions seem simple, but several sources of error can cause misunderstanding. This article discusses the logical shortcomings and misinterpretations of these representations through the example of convex quadrilaterals. Mathematical models used in the analysis can be discovered in the flowcharts ambiguities.

Keywords: hierarchical concept structures, flowchart, convex quadrilateral

Hierarchical system

The hierarchical connections of a concept system are often presented with graphs. Various methods are used to illustrate a hierarchical system, which this article intends to demonstrate on an example of structure of convex quadrilaterals.

The mathematical models of the hierarchical knowledge representation set out from the basic set $\mathcal{C}$. The elements of the basic set $\mathcal{C}$ can be the "knowledge units". The knowledge units can be modelled by the classes or by the structured objects. Also the elements of the set $\mathcal{C}$ are called pre-concepts. In this article we demonstrate it on an example of the structure of convex quadrilaterals that can be seen in Figure 1. (The expression of general quadrilateral means a quadrilateral without special characteristics.)

In general a hierarchy of concepts exists on set $\mathcal{C}$ and it is based on an order relation. One order relation can be either a weak partial order relation (reflexive, transitive, antisymmetric) or a strict partial order relation (irreflexive, transitive and asymmetric). From the point of view of didactics, the containment between concepts is primary. There are many conceptions about the formal basic containment relation $\leq$ on the set $\mathcal{C}$. In this paper, we take the binary concept relation $\leq$ for representing this containment relation of the concepts. The set $\mathcal{C}$ of pre-concepts and the relation $\leq$ create a formal concept system among these forms.

We are going to use a *hierarchical system* as the tuple $H := (\mathcal{C}, \leq)$ where $\mathcal{C}$ is a set whose elements are called pre-concepts, and $\leq$ is a binary partial order relation on the set $\mathcal{C}$ (i.e. $\leq \subseteq \mathcal{C} \times \mathcal{C}$ is a reflexive, transitive, antisymmetric relation).

The examined convex quadrilaterals C			
Figure as object	code	Figure as object	code
convex quadrilateral	1	parallelogram	7
general quadrilateral	2	rhombus	8
trapezium	3	rectangle	9
orthogonal trapezium	4	square	10
symmetric trapezium	5	cyclic quadrilateral	11
kite	6	circumscribed quadrilateral	12

Figure 1: The examined convex quadrilaterals with code

The finite hierarchical system can be given by containment table: we write a rectangular table with one row for each pre-concept and one column for each pre-concept, having a ≤ in the intersection of row *g* with column *m* if (sub)-concept *g* is (super)-concept *m* as representation of the knowledge units. The finite hierarchical system can be visualized by a graph with Hasse-convention.

Olosz (2005) gives one concrete inquest through the system of symmetric quadrilaterals. The related containment table is presented in figure 2/a and the graph of this inquest is presented in figure 2/b (Teaching of quadrilaterals in Hungarian elementary schools adopts this construction.)

Containment table			quadrilaterals					
			4	6	7	8	9	10
quadrilaterals	symmetric trapezium	5	≤					
	kite	6		≤				
	parallelogram	7			≤			
	rhombus	8		≤	≤	≤		
	rectangle	9	≤		≤		≤	
	square	10	≤	≤	≤	≤	≤	≤

Figure 2/a: The formal containment table '≤' of symmetric quadrilaterals

The examination of the quadrilateral classes is completed with the additional quadrilateral varieties. *Mitroica (1987)* gives one concrete hierarchical structure diagram of convex quadrilaterals that can be seen in Figure 3/a. (The containment table of this graph is presented in Figure 3/b.)

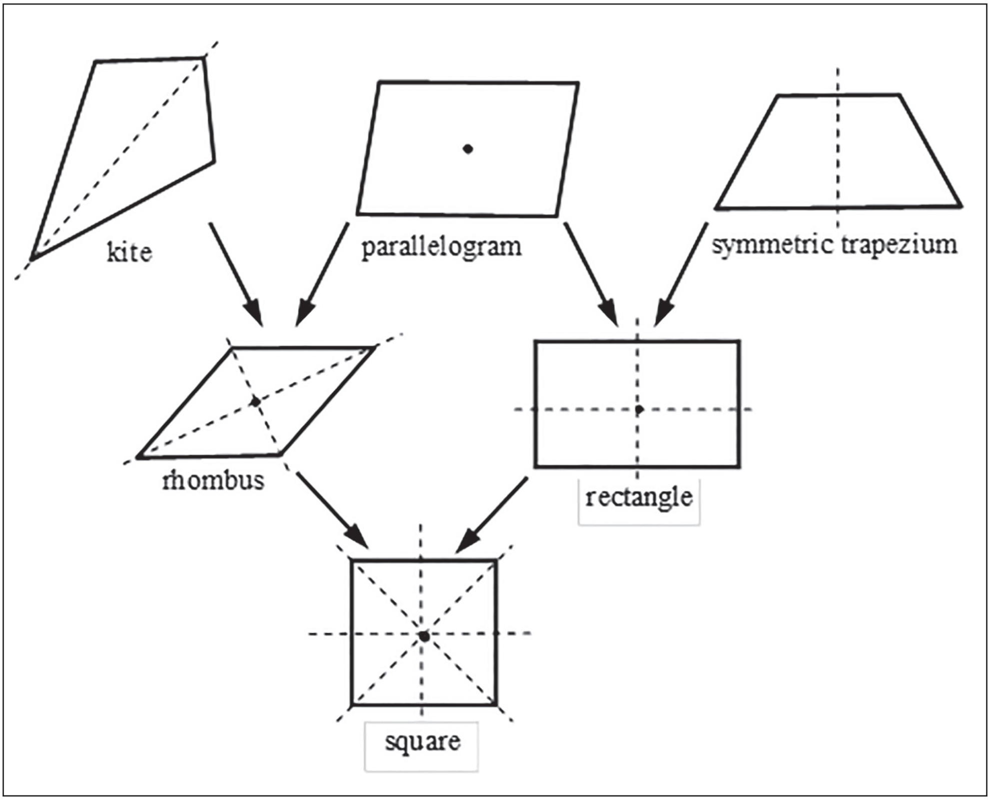


Figure 2/b: The genesis diagram of symmetric quadrilaterals by Olosz (p60)

It's important to observe that this analysis characterizes the examined classes of quadrilaterals only from containment aspect. One of the formal failures of the diagram is that it can lead to false conclusions, because it can be seen from the figure, that the common part of the sets of circumscribed quadrilateral and trapezium is the rhombus i.e.

$$\textit{circumscribed quadrilaterals} \cap \textit{trapeziums} = \textit{rhombuses.}$$

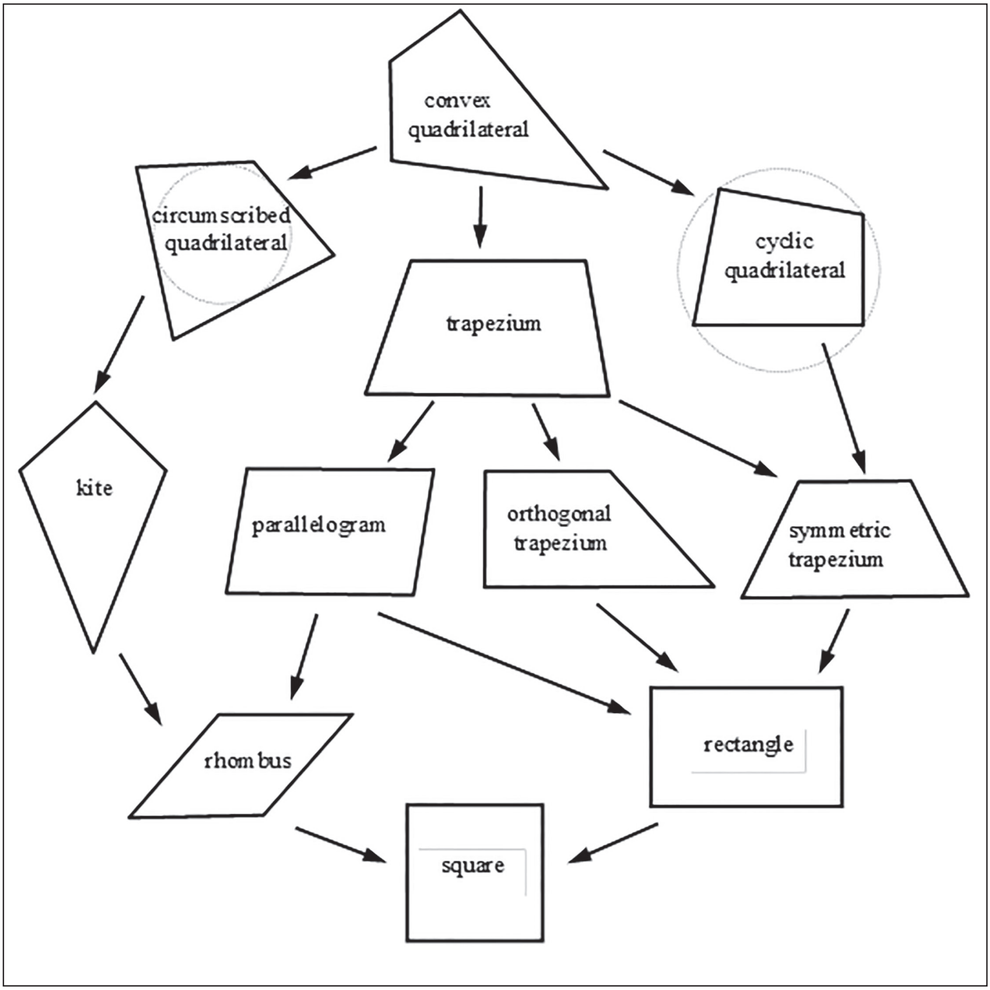


Figure 3/a: The hierarchical structure of the convex quadrilaterals by Mitroica (p58)

This description of the concept has one relevant drawback from the point of view of didactics: this formal model of the pre-concept (as an element of basic set) does not comprehend the objects and the attributes of the concept directly, i.e. a pre-concept has only intuitive content with explicit denomination. As a consequence, this graph does not present the aspect of the concept generalising or specialising, that is a central aspect in teaching.

containment table			**quadrilaterals $\mathcal{C}$**										
			1	3	4	5	6	7	8	9	10	11	12
quadrilateral $\mathcal{C}$	convex quadrilateral	1	≤										
	trapezium	3	≤	≤									
	orthogonal trapezium	4	≤	≤		≤							
	symmetric trapezium	5	≤	≤	≤							≤	
	kite	6	≤				≤						≤
	parallelogram	7	≤	≤				≤					
	rhombus	8	≤	≤			≤	≤	≤				≤
	rectangle	9	≤	≤	≤	≤		≤		≤		≤	
	square	10	≤	≤	≤	≤	≤	≤	≤	≤	≤	≤	≤
	cyclic quadrilateral	11	≤									≤	
	circumscribed quadrilateral	12	≤										≤

Figure 3/b: The formal containment table of convex quadrilaterals

The content of the convex quadrilaterals

To characterize the convex quadrilaterals the examined features need to be fixed. The mathematical modelling of the characterization sets out from the attributes set $\mathcal{A}$. This paper is based on the Hungarian National Curriculum *[NAT,1995]*, in which the constructions of curriculum are based on symmetrical and metrical properties. The symmetrical and metrical attributes are among the examined features. The attributes of set $\mathcal{A}$ with the code of the attributes are presented in figure 4.

The set of the attributes: $\mathcal{A}$	
Attributes of quadrilaterals	code
(at least) one parallel pair of sides	A
(at least) one equal pair of sides	B
two parallel pairs of sides	C
two equal pairs of sides	D

The set of the attributes: $\mathcal{A}$	
Attributes of quadrilaterals	code
equal sides	E
equal diagonals	F
orthogonal diagonals	G
diagonals bisect each other	H
(at least) one equal pair of angles	I
two equal pairs of angles	J
equal angles	K
central symmetry	L
(at least) one axisymmetry about a diagonal	M
(at least) one axisymmetry about a perpendicular bisectional line	N
two axisymmetries about diagonals	O
two axisymmetries about perpendicular bisectional lines	P
axisymmetry about a diagonal and about a perpendicular bisectional line	Q
circumscribed circle	R
inscribed circle	S

Figure 4: The set of the attributes with code

The cross table of the convex quadrilaterals

The relation $\mathcal{R}$ can be defined between objects set and attributes set on a natural way, that can be written down with a cross table in the simplest format (Figure 5 presents one cross table of the convex quadrilaterals). A cross table is a rectangular table, with one row for each object (the examined quadrilateral class) and one column for each attribute (the examined features of quadrilaterals) per convention. The intersection of row *o* with column *a* contains the information with cross sign whether object *o* has attribute *a.*

It is important to observe that this analysis characterizes the examined classes of quadrilaterals only from containment aspect, i.e.:

– the cross sign in the intersection of row *o* with column *a* means that all elements of quadrilateral class *o* have the attribute *a*,

– the missing cross sign in the intersection of row *o* with column *a* means that *(at* *least) one element* exists in quadrilateral class *o*, that does not have the attribute *a*, i.e. this should not be taken to mean that all elements of the quadrilateral class *o* do not have the feature *a*.

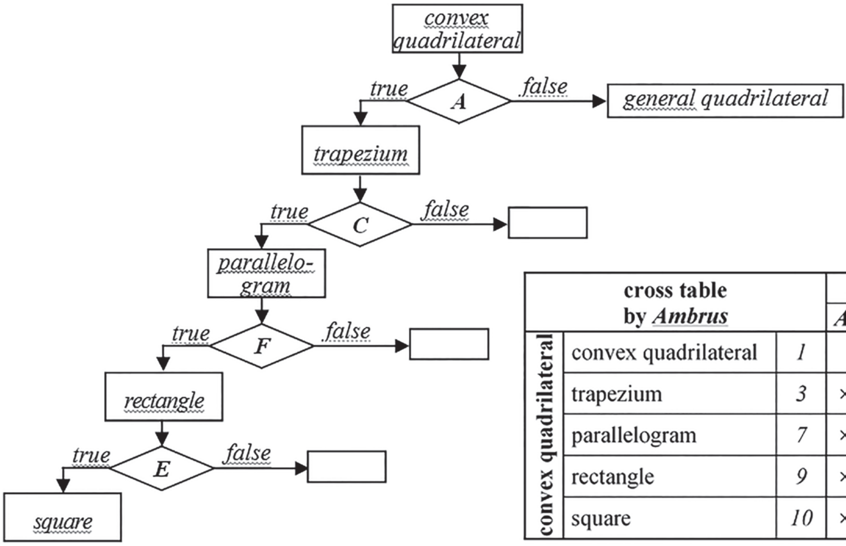
CROSS TABLE $\mathcal{R}$			attribute $\mathcal{A}$																		
			A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S
quadrilateral $\mathcal{C}$	convex quadrilateral	1																			
	general quadrilateral	2																			
	trapezium	3	×																		
	orthogonal trapezium	4	×								×										
	symmetric trapezium	5	×	×				×			×	×				×				×	
	kite	6		×		×			×		×				×						×
	parallelogram	7	×	×	×	×				×	×	×		×							
	rhombus	8	×	×	×	×	×		×	×	×	×		×	×		×				×
	rectangle	9	×	×	×	×		×		×	×	×	×	×		×		×		×	
	square	10	×	×	×	×	×	×	×	×	×	×	×	×	×	×	×	×	×	×	×
	cyclic quadrilateral	11																		×	
	circumscribed quadrilateral	12																			×

Figure 5: The cross table $\mathcal{R}$ of convex quadrilaterals by education plan of Hungary

Producing a cross table is a useful starting-point in planning education. The construction of the cross table (the circumscribing of basic sets of objects and attributes, filling of table) requires the mathematical and also the mathematicdidactical competent simultaneously.

The flowchart of the convex quadrilaterals

Plato in the *Sophist* gives one method of the concept definition: The extension of the concept has to be narrowed down with the distinctive attribute step by step. *Ambrus (1995)* presents Plato's idea precisely through the method of specialization of the convex quadrilaterals, i.e. narrowing down the extension of the concept through the intension of the concept. This classical method puts to use the relevant attribute as leading feature (*differentia specific*).



	cross table by *Ambrus*		attribute $\mathcal{A}$			
			A	*C*	*E*	*F*
convex quadrilateral	convex quadrilateral	*1*				
	trapezium	*3*	×			
	parallelogram	*7*	×	×		
	rectangle	*9*	×	×		×
	square	*10*	×	×	×	×

Figure 6: The hierarchical systematisation of the convex quadrilaterals by Ambrus (p. 60)

Attribute *A* (*parallel pair of sides)* divides the set $\mathcal{C}$ (convex quadrilaterals) into two parts. The first subset (trapezium) has the relevant attribute *A*, the second subset (general quadrilateral) is the complementary set of the first subset, i.e. the elements of the second subset do not have the attribute *A*. The first set (trapezium) can be repeatedly divided in two parts by attribute *C (two parallel pairs of sides)*, etc. This diagram really describes one decision algorithm. Figure 6 presents the diagram of the method by *Ambrus*.

This illustration very accurately depicts the structure of the explicit definitions; it helps to learn the forms of the explicit definition. (For example: The square is a rectangle with attribute *E*, i.e. with equal sides.) The concept is fixed from the next genus (*genus proximum*) with the distinctive attribute (*differentia specific*) in the explicit definition. Important to note that this method has a formal defect, because it can lead to having false conclusion: the next genus (*genus proximum*) exists unanimously. The square can be derived from rhombus, i.e. the square is a rhombus with attribute *K*, i.e. with equal angles.

The defect of the diagram is that the rhombus can not be embedded in this explicit definition chain. The integration method of the rhombus and other convex quadrilateral classes in the diagram has to be generalised.

Pelle (1974) gives an algorithm by which the objects of the quadrilateral classes can be sorted. It can be seen in Figure 7/a. This flowchart describes a method, by which it can be decided if a concrete quadrilateral belongs to one special class of convex quadrilaterals. The activity of the decision (the ranging and the preclusion) is the first function in this method. The decisions are based on the formal logic considerations of the relevant attributes. The relevant attributes can be selected from the set of attributes *E* in several ways, so other flowcharts can also be constructed from a cross table.

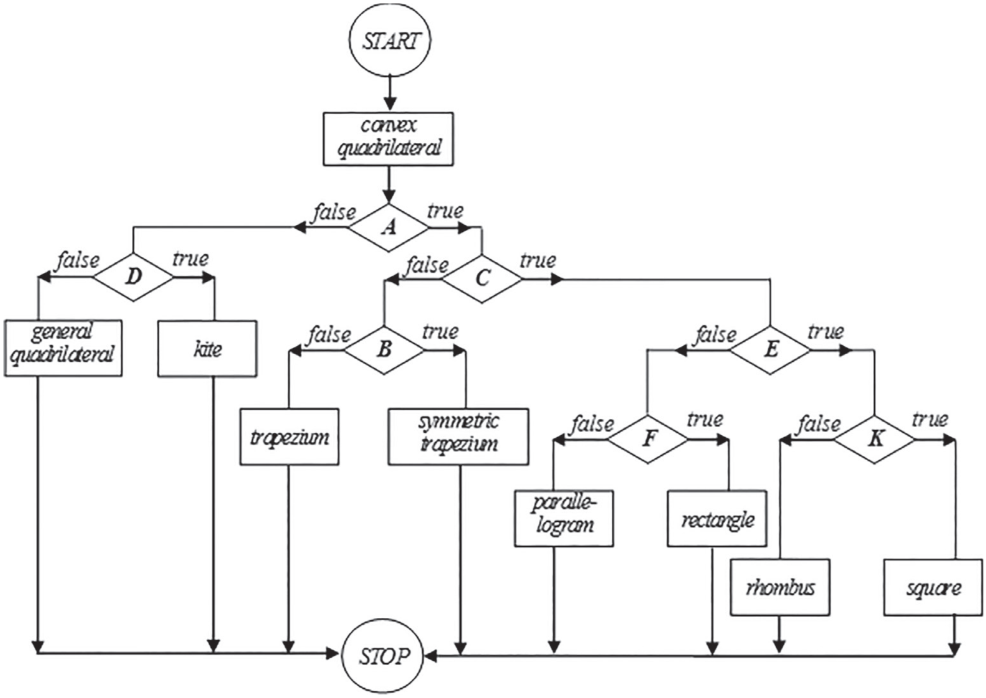


Figure 7/a: The process diagram to categorize the convex quadrilaterals by Pelle

Figure 7/b presents the selected attributes and the used information (with × sign) of cross table 5. In general the construction of an algorithm does not need to employ all information of the cross table. Table 7/b presents also the unused information (as lost information with – sign) of the cross table.

Taking them each individually, it can be read from this algorithm that a concrete class of the quadrilaterals does not have a given attribute. (For example: The kite does not have the attribute *A*, i.e. it does not have parallel pair of sides.) Figure 7/b contains the information from not only true but also false attributes by Figure 7/a. (Table 7/b presents also this misleading information with + sign.) This additional implicit information may lead to a false conclusion, because it can be seen from Figure 7/b, that a kite '*never*' has (at least) one parallel pair of sides (see cell *A6* in the table 7/b), but a rhombus 'always' has at least one pair of parallel sides (see cell *A8* in the table 7/b), i.e. the false conclusion can be drawn:

a rhombus is not a kite.

The problem appears that a net of concept cannot simply be described with a flow-chart.

INFORMATION TABLE OF FIGURE 7.1			attribute						
			A	B	C	D	E	F	K
quadrangles	convex quadrilateral	1							
	general quadrilateral	2	+			+			
	trapeze	3	×	+	+				
	symmetric trapeze	4	×	×	+			–	
	kite	6	+	–		×			
	parallelogram	7	×	–	×	–	+	+	
	rhombus	8	×	–	×	–	×		+
	rectangle	9	×	–	×	–	+	×	–
	square	10	×	–	×	–	×	–	×

Markings:
×: used information
+: misleading information
–: lost information

Figure 7/b: The information table of Figure 7/a

When preparing such a figure one should pay attention to these errors, because they cause confusions. Right figures are prepared with tools of the formal concept analysis and of trichotomyc concept model. (It must not be forgotten what is illustrated: the relationships between real objects or between their names.)

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Fogalmi hierarchiák klasszikus ábrázolásai gráfokkal

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A fogalmi hierarchiákat gyakorta gráfok szemléltetik. E tipikus ábrázolások vagy tartalmazási, vagy származtatási kapcsolatra épülnek. A klasszikus vizuális ábrázolások egyszerűnek tűnnek, de több hibaforrásuk is félreértést okozhat. Ez a cikk ezen ábrázolások hiányosságait, logikai félreértelmezhetőségét mutatja be a konvex négyszögek példáján keresztül. Az elemzésben használt matematikai modellel feltárhatók a folyamatábrák kétértelműségei is.

Kulcsszavak: hierarchikus fogalmi struktúra, folyamatábra, konvex négyszög